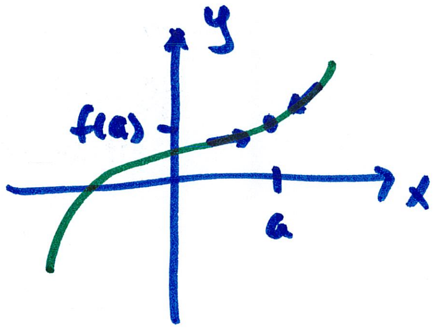
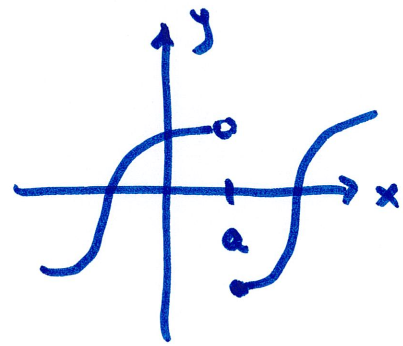
15.2 Limit and Continuity

recall that if $\lim_{x \to a} f(x) = L$ means we can make $f(x)$ as close to L as we want by making x sufficiently close to a

if the limit exists at $x = a$, then it doesn't matter how we approach a

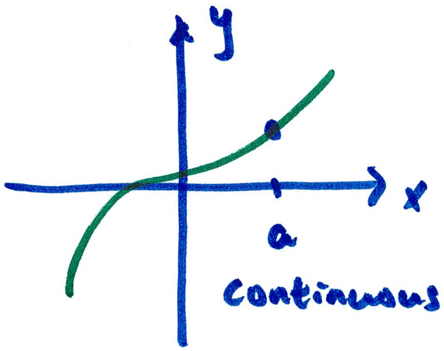
$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a} f(x) = L$$



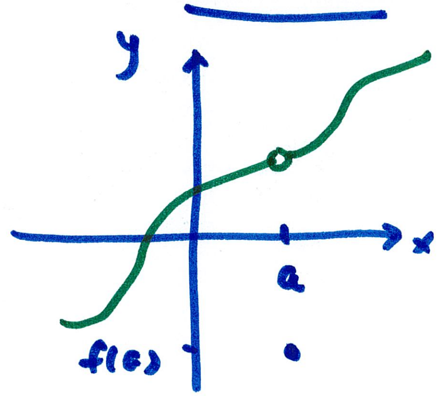


limit DNE because $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

moreover, if $\lim_{x \to a} f(x) = f(a)$ then $f(x)$ is continuous



continuous at $x = a$



NOT continuous at $x = a$

we know many types of functions are continuous

polynomial, sine, cosine, exponentials $\to$ continuous everywhere

rational, logarithmic $\to$ continuous wherever defined

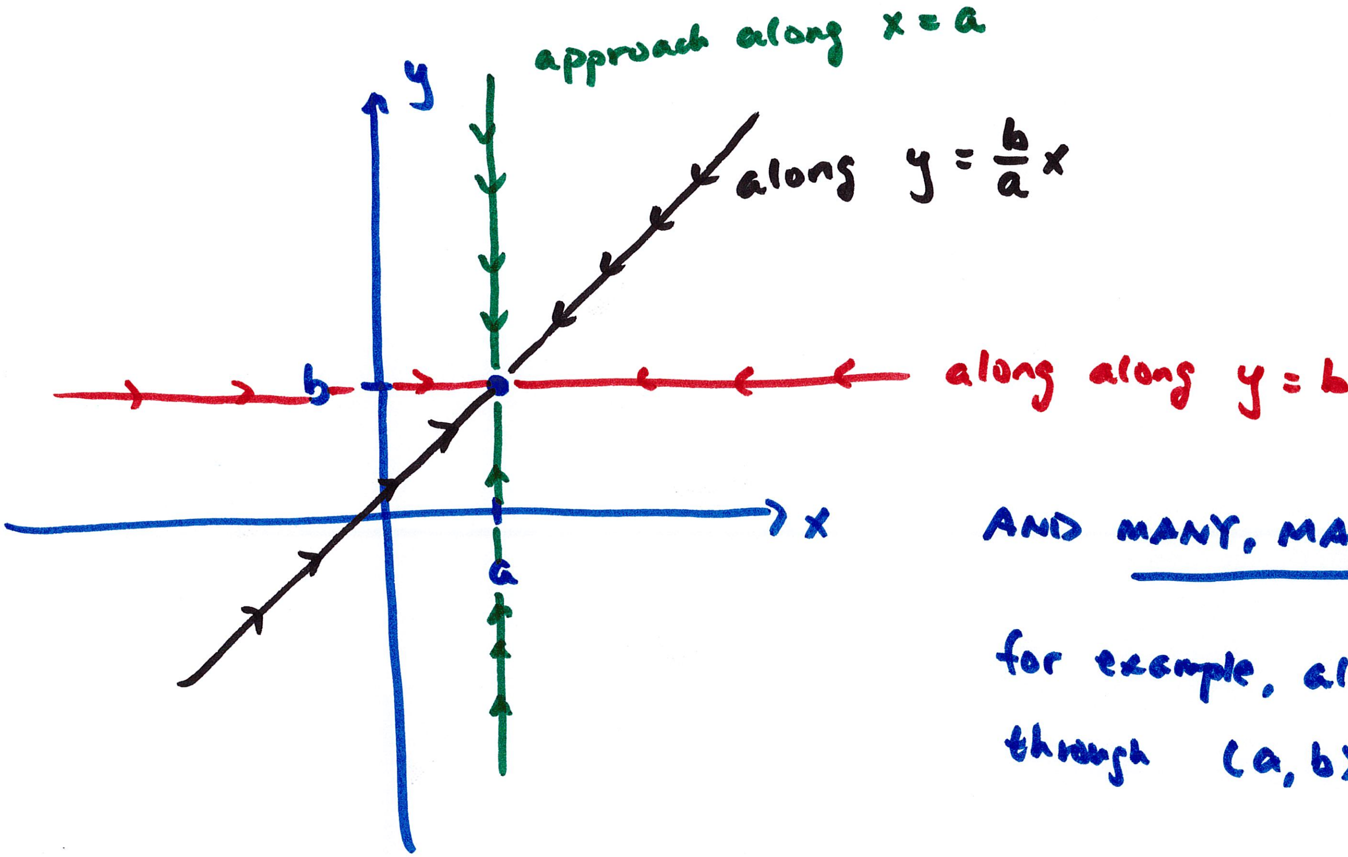
so, if we know that a function is continuous at $x = a$, then

finding the limit is easy: $\lim_{x \to a} f(x) = f(a)$

Almost everything we know about limits and continuity from one-variable calculus carry over to multivariate calculus

$\lim_{(x,y) \to (a,b)} f(x,y) = L$ means we can make $f(x,y)$ as close to L as we want by making (x,y) close to (a,b) as needed

but <u>how</u> we approach (a,b) is more complicated



AND <u>MANY, MANY</u> more possibilities

for example, along a parabola through (a, b)

if $\lim_{(x,y)\to(a,b)} f(x,y)$ <u>exists</u>, then the path does <u>NOT</u> affect the limit

(if two paths lead to different values, then limit does not exist)

example $$\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right)$$

before worrying about the approach path, let's use continuity to see if we can find the limit

$\ln\left(\frac{1+y^2}{x^2+xy}\right)$ is logarithmic, so is continuous at (a,b) if it is defined at (a,b)

is $\ln\left(\frac{1+y^2}{x^2+xy}\right)$ defined at $(1,0)$?

yes, $\ln\left(\frac{1+0}{1+0}\right) = \ln(1) = 0$ is defined

therefore, the function is continuous at $(1,0)$

so $$\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right) = \ln\left(\frac{1+0}{1+0}\right) = \ln(1) = 0$$

example $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2-4x^2}{2x^2+y^2}$

$\dfrac{y^2-4x^2}{2x^2+y^2}$ is rational and is continuous at (a,b) if defined at (a,b)

defined at $(0,0)$? no, $\frac{0}{0} \to$?

but the limit may still exist at $(0,0)$

for the limit to exist, ALL paths must lead to the same limit

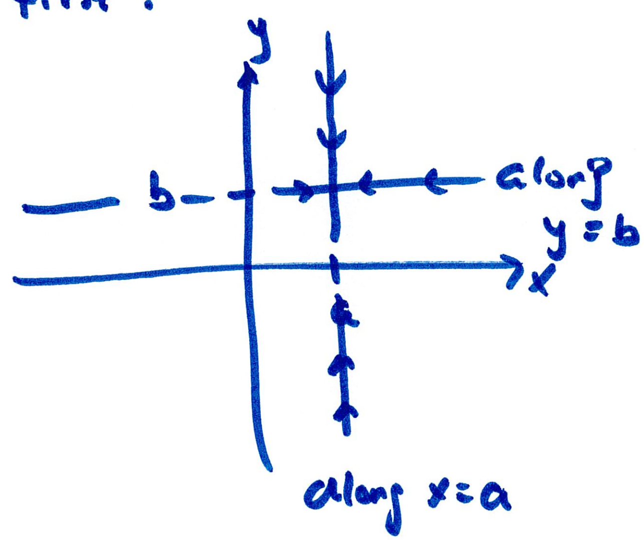
$\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2-4x^2}{2x^2+y^2}$

a b

check the easiest two first :

along $x=a$

along $y=b$

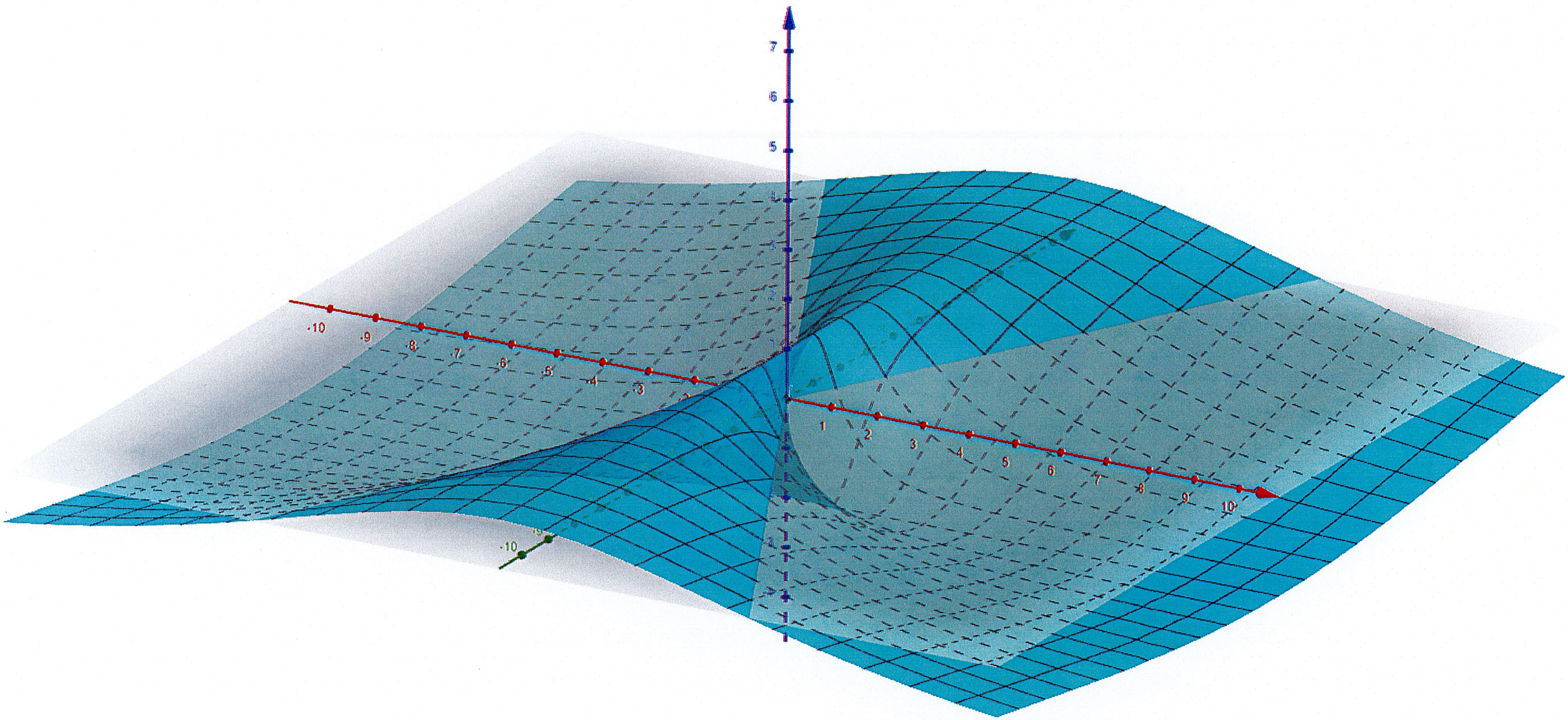


along $x=a=0$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2-4x^2}{2x^2+y^2} = \lim_{y\to 0} \frac{y^2-0}{0+y^2} = 1$

along $y=b=0$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2-4x^2}{2x^2+y^2} = \lim_{x\to 0} \frac{0-4x^2}{2x^2+0} = -2$

they are not equal, so not ALL paths lead to same limit, so

$$\lim_{(x,y)\to(0,0)} \frac{y^2-4x^2}{2x^2+y^2} \text{ DNE}$$



example $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y}$

clearly $\dfrac{x^2-y^2}{x+y}$ is not defined at $(0,0)$

along $x=0$: $\displaystyle\lim_{y\to 0} \frac{-y^2}{y} = \lim_{y\to 0} -y = 0$

along $y=0$: $\displaystyle\lim_{x\to 0} \frac{x^2}{x} = 0$

BUT this IS NOT enough!

along $y=x$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{x\to 0} \frac{x^2-x^2}{x+x} = \lim_{x\to 0} \frac{0}{2x} = 0$

never 0

Still not enough

what's next? $y=x^2$? $y=x^3$? $y=e^x-1$? $y=\sin x$?

think of another way to find limit that does NOT ~~depend on~~ assume a particular path one way: table of values w/ $(x, y) \to (0,0)$

another way: algebraically ~~mod~~ simplify the expression

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{\cancel{(x+y)}(x-y)}{\cancel{x+y}}$$

$$= \underbrace{\lim_{(x,y)\to(0,0)}}_{\text{does not depend on path}} (x-y) = 0$$

does NOT assume any particular path

so, the limit is 0

$x - y \to 0$ as $x \to 0$, $y \to 0$ no matter how example

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+2y^2)}{x^2+2y^2}$$

let $u = x^2+2y^2$ then $(x,y)\to(0,0)$ means $u\to 0$

so the limit becomes $\displaystyle\lim_{u\to 0} \frac{\sin(u)}{u} = 1$

true w/o assuming any path

so limit is 1.