15.3 Partial Derivatives

recall if $y = f(x)$ then the derivative of y with respect to x is

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

this is also the rate of change of y as x changes.

for a function of two variables, $z = f(x,y)$, z is affected by <u>both</u> x and y. for example, the wind chill is affected by the air temperature and wind speed

$$w = w(t, v)$$

to better understand how each variable affects the function, we want to isolate its effect on the function

→ partial derivative (how function changes as one variable changes while the other is constant)

$z = f(x, y)$

the partial derivative of $z = f(x,y)$ with respect to x is

$$\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

y is NOT changing so is treated as a constant

funny looking d

"live variable" – the one that is changing

the partial derivative of $z = f(x,y)$ with respect to y is

$$\frac{\partial f}{\partial y} = f_y = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

x is NOT changing so is treated as a constant

f_x : how f is changing due to x alone

f_y : " " " y alone

in practice, we rarely use the limit definition

we just use the various rule we know while remembering which variable is held constant

example $f(x,y) = x^2 + y^3 + xy$

the partial with respect to x is y is constant

$$\frac{\partial}{\partial x}(x^2+y^3+xy) = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^3) + \underbrace{\frac{\partial}{\partial x}(xy)}_{\text{treat like } \frac{\partial}{\partial x}(\text{constant}\cdot x)}$$

$$= 2x + 0 + y = \boxed{2x+y}$$

the partial with respect to y is x is constant

$$\frac{\partial}{\partial y}(x^2+y^3+xy) = \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial y}(xy)$$

$$= 0 + 3y^2 + x = \boxed{3y^2+x}$$

example $z = f(x,y) = x^3 \tan(xy)$

y is constant $\quad \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\left(x^3 \tan(xy) \right)$ product rule

$$= x^3 \cdot \underbrace{\frac{\partial}{\partial x}\tan(xy)} + \tan(xy)\cdot \frac{\partial}{\partial x} x^3$$

$$= x^3 \cdot \sec^2(xy)\cdot y + \tan(xy)\cdot 3x^2 = \boxed{x^3 y \sec^2(xy) + 3x^2 \tan(xy)}$$

from chain rule

$\frac{\partial}{\partial x}(xy)$

constant

x is constant $\quad \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left(x^3 \cdot \tan xy \right) = x^3 \cdot \dfrac{\partial}{\partial y}(\tan xy) = x^3 \cdot \sec^2(xy)\cdot \underbrace{\dfrac{\partial}{\partial y}(xy)}_{x}$

$$= \boxed{x^4 \sec^2(xy)}$$

we can take partial derivatives of partial derivatives

<u>example</u> $f(x,y) = e^x \sin y$

y is constant $\frac{\partial f}{\partial x} = f_x = e^x \sin y$

x const $\frac{\partial f}{\partial y} = f_y = e^x \cos y$

second order partials:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}(e^x \sin y) = e^x \sin y$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = \frac{\partial}{\partial y}(e^x \cos y) = -e^x \sin y$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\underbrace{\partial y \partial x}_{\text{order is right to left}}} = f_{xy} = \frac{\partial}{\partial y}(e^x \sin y) = e^x \cos y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = \frac{\partial}{\partial x}(e^x \cos y) = e^x \cos y$$

Same
NOT a
coincidence

note $f_{xy} = f_{yx}$

it turns out that these "mixed partials" are always equal

~~if~~ where $f(x,y)$ is defined and f_x and f_y are continuous

<u>example</u> $f(x,y) = e^{x^2 y}$

$$f_x = 2xye^{x^2 y}$$

$$f_y = x^2 e^{x^2 y}$$

product rule $f_{xy} = \frac{\partial}{\partial y}(2xy \cdot e^{x^2 y}) = 2xy \cdot \frac{\partial}{\partial y}(e^{x^2 y}) + e^{x^2 y} \cdot \frac{\partial}{\partial y}(2xy)$

$$= 2xy \cdot x^2 e^{x^2 y} + e^{x^2 y} \cdot 2x = \boxed{2xe^{x^2 y}(x^2 y + 1)}$$

prod. rule $f_{yx} = \frac{\partial}{\partial x}(x^2 e^{x^2 y}) = x^2 \cdot e^{x^2 y} \cdot 2xy + e^{x^2 y} \cdot 2x$

$$= \boxed{2xe^{x^2 y}(x^2 y + 1)}$$

(Clairaut's Theorem)

the equality of mixed partials is true for functions of higher number of variables

<u>example</u> $f(x,y,z) = xyz$

$f_x = yz \longrightarrow \frac{\partial}{\partial x}(xyz) = yz$

$f_y = xz$

$f_z = xy$

$f_{xy} = z$ $\qquad f_{yx} = z$

$f_{xyz} = 1$

$f_{yxz} = 1$

$f_{zyx} = 1$