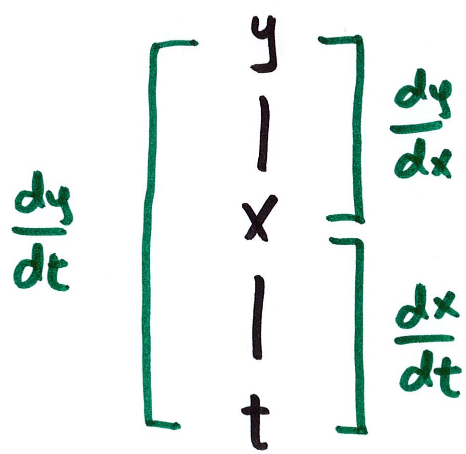
15.4 The Chain Rule

recall if $y = f(x)$ and $x = g(t)$ then to find how y changes with respect to t, ~~or~~ we use the Chain Rule

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \qquad \text{or} \qquad y' = f'(g(t))\,g'(t)$$

to get a better understanding of this and to extend to functions of more variables, let's look at how variables depend on one another



each step down this tree is a derivative

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

for functions of more variables, we step down the tree the same way,
but if there is a fork we use partial derivative.

example

$z = f(x, y) = x + y^2$

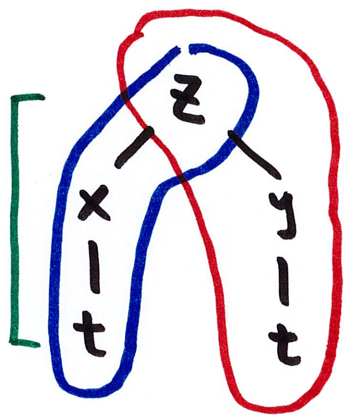
$x = e^t$, $y = \ln t$

ultimately, z is a function of t

$z = g(t)$ $\quad \frac{dz}{dt} = ?$

tree:

$\frac{dz}{dt}$



step down a fork: partial derivative

no fork: regular

$\frac{dz}{dt}$ contains the $\underbrace{\text{effect of } t \text{ on } x}_{\text{blue branch}}$ and $\underbrace{t \text{ on } y}_{\text{red branch}}$

$$\frac{dz}{dt} = \underbrace{\frac{\partial z}{\partial x}\frac{dx}{dt}}_{\text{blue}} + \underbrace{\frac{\partial z}{\partial y}\frac{dy}{dt}}_{\text{red}}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$z = x + y^2$

$x = e^t$

$y = \ln t$

$$\frac{dz}{dt} = (1)(e^t) + (2y)\left(\frac{1}{t}\right) = e^t + 2(\ln t)\left(\frac{1}{t}\right) = \boxed{e^t + \frac{2\ln t}{t}}$$

represents the variable of interest so write answer in terms of it

verify by subbing out x, y

$$z = x + y^2 = e^t + (\ln t)^2$$

$$\frac{dz}{dt} = e^t + 2(\ln t)\frac{1}{t} = \boxed{e^t + \frac{2\ln t}{t}}$$ same

Substitution may not always be practical (e.g. x, y are complicated or implicit functions of t).

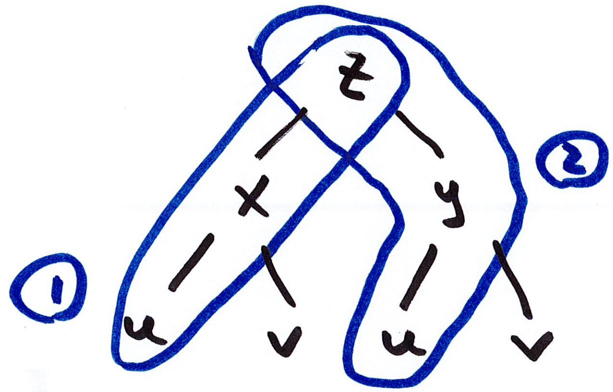
example $z = \sin(x+y)$

$x = u^2 + v$

$y = 1 - 2uv$

ultimately, z is a function of u, v

find: $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$



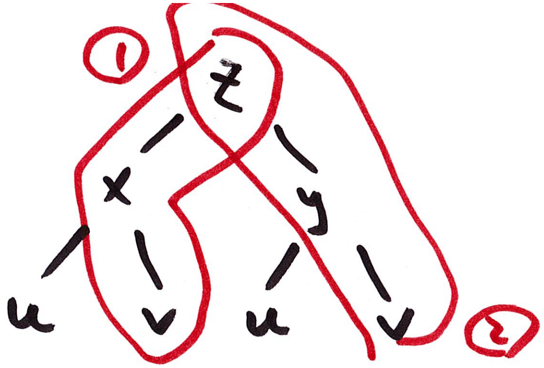
$$\frac{\partial z}{\partial u} = \underbrace{\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}}_{1} + \underbrace{\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}}_{2}$$

$$\frac{\partial z}{\partial u} = \cos(x+y) \cdot 2u + \cos(x+y) \cdot (-2v)$$

write in terms of u

$$= \cos(u^2+v+1-2uv) \cdot 2u + \cos(u^2+v+1-2uv)(-2v)$$

$$= \boxed{2(u-v)\cos(u^2+v+1-2uv)}$$



$(1-2u)\cos(u^2+v+1-2uv)$

factored. sub u, v in.

$$\frac{\partial z}{\partial v} = \underbrace{\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}}_{1} + \underbrace{\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}}_{2} = \cos(x+y)(1) + \cos(x+y)\cancel{(1-2u)}(-2u)$$

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now we apply chain rule to implicit differentiation

$$x^2+2y^2=4$$

if y is an implicit function of x, find $\frac{dy}{dx}$

old way: $\frac{d}{dx}(x^2+2y^2) = \frac{d}{dx}(4)$

$$2x + 4y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \cancel{-\frac{4y}{2x} = -\frac{2y}{x}} = \frac{-2x}{4y} = -\frac{x}{2y}$$

now let's see this from the chain rule perspective

$$x^2 + 2y^2 = 4$$

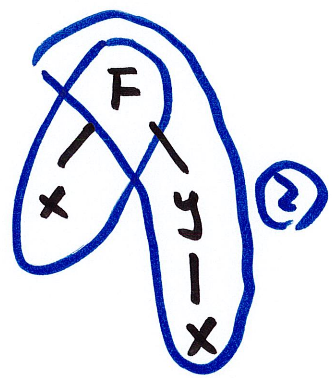
$y =$ some function of x

define $F(x,y) = x^2 + 2y^2 - 4 = 0$

but since y is a function of x,

$F(x,y)$ is really some function of x, too. Call it $f(x)$

$$f(x) = \underbrace{F(x,y)} = 0$$



want this ①

so, $$\frac{df}{dx} = \underset{①}{\frac{\partial F}{\partial x}} + \underset{②}{\frac{\partial F}{\partial y}}\boxed{\frac{dy}{dx}} = 0$$

because $f(x) = F(x,y) = 0$

so its deriv. is also 0.

solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-F_x}{F_y} = -\frac{2x}{4y} = \boxed{-\frac{x}{2y}}$$

same

we can do the same w/ functions of more variables

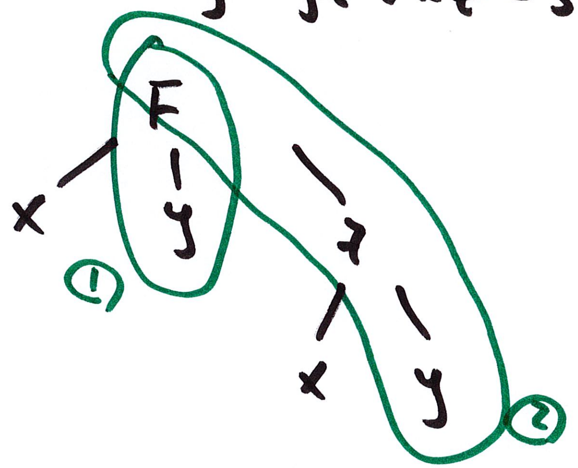
<u>example</u> $xy + yz + xz = 3$

z is some implicit function of x and y

find $\dfrac{\partial z}{\partial y}$

because z is some function of x, y

define $F(x,y,z) = xy + yz + xz - 3 = 0 = \overbrace{f}(x,y)$



$f(x,y) = F(x,y,z) = 0$

want!

$$\frac{\partial f}{\partial y} = \underset{①}{\frac{\partial F}{\partial y}} + \underset{②}{\frac{\partial F}{\partial z}\boxed{\frac{\partial z}{\partial y}}} = 0 \rightarrow \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = \boxed{-\frac{x+z}{y+x}}$$