15.5 Directional Derivative and the Gradient

(NOT on exam 1)

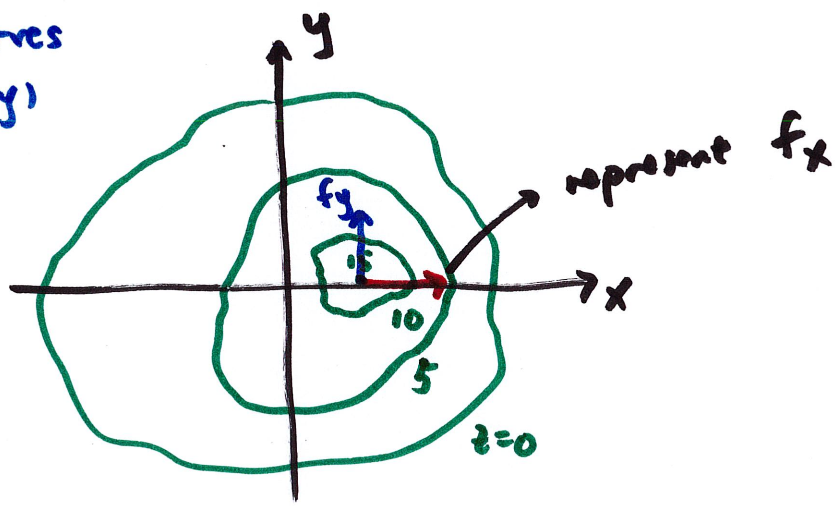
$z = f(x,y)$

we know $\frac{\partial f}{\partial x} = f_x$ is the rate of change of $f(x,y)$ as x changes w/ y held constant

another interpretation: how fast is the height of surface $z=f(x,y)$ changing as we move on the surface in direction parallel to x-axis

same idea for $\frac{\partial f}{\partial y} = f_y$

level curves of $z=f(x,y)$



f_x: how fast height changes if moving east

f_y: how fast height changes if moving north

what about if we move in a direction not parallel to x-axis or y-axis (e.g. north east?)

the ~~d~~ Directional Derivative gives us that information

let $\vec{u} = \langle a, b \rangle$ be a unit vector then the rate of change of $f(x,y)$ in that direction is

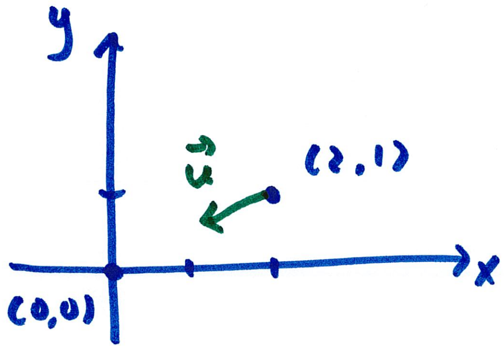
$$\boxed{D_{\vec{u}} f(x,y) = \lim_{h \to 0} \frac{f(x+ha, y+hb) - f(x,y)}{h}}$$

the practical form is

$$\boxed{D_{\vec{u}} f(x,y) = \frac{\partial f}{\partial x}(x,y)\, a + \frac{\partial f}{\partial y}(x,y)\, b = f_x(x,y)\, a + f_y(x,y)\, b}$$

where $\vec{u} = \langle a, b \rangle$ is a unit vector example $f(x,y) = \cos(2x+3y)$

find the directional derivative at $(2,1)$ in the direction toward origin



find the unit vector $\vec{u}$ from $(2,1)$ toward $(0,0)$

$$\vec{u} = \frac{\langle -2, -1 \rangle}{|\langle -2, -1 \rangle|} = \left\langle \underset{a}{\frac{-2}{\sqrt{5}}}, \underset{b}{\frac{-1}{\sqrt{5}}} \right\rangle$$

$$D_{\vec{u}} f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$$

$$= -2\sin(2x+3y) \cdot \frac{-2}{\sqrt{5}} - 3\sin(2x+3y) \cdot \frac{-1}{\sqrt{5}}$$

$$D_{\vec{u}} f(2,1) = -2\sin(7) \cdot \frac{-2}{\sqrt{5}} - 3\sin(7) \cdot \frac{-1}{\sqrt{5}} = \boxed{\frac{7}{\sqrt{5}} \sin(7)}$$

this is how fast $f(x,y) = \cos(2x+3y)$ changes at $(2,1)$ in the direction toward $(0,0)$

back to $D_{\vec{u}} f = f_x\, a + f_y\, b$

rewrite it as $= \underbrace{\langle f_x, f_y \rangle}_{\text{this vector is called the gradient}} \cdot \underbrace{\langle a, b \rangle}_{\vec{u} \text{ that gives direction}}$

$$\boxed{\text{gradient: } \vec{\nabla} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \frac{\partial f}{\partial x}(x,y)\,\vec{i} + \frac{\partial f}{\partial y}(x,y)\,\vec{j}}$$

"del"

$$\boxed{\vec{\nabla} f(x,y,z) = \langle f_x, f_y, f_z \rangle}$$

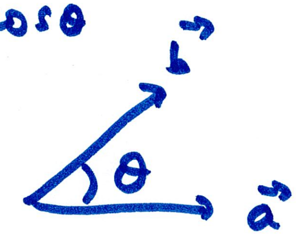
the directional derivative is now also

$$\boxed{D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}}$$

what do the magnitude and direction of $\vec{\nabla} f$ tells us?

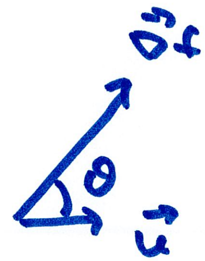
$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$



$$= |\vec{\nabla} f| \underbrace{|\vec{u}|}_{1} \cos\theta$$

because $\vec{u}$ is a unit vector



$$D_{\vec{u}} f = |\vec{\nabla} f| \cos\theta \qquad -1 \leq \cos\theta \leq 1$$

therefore, the ~~derivati~~ directional derivative has a maximum value equal to the magnitude of the gradient.

— this happens when $\vec{\nabla} f \parallel \vec{u}$

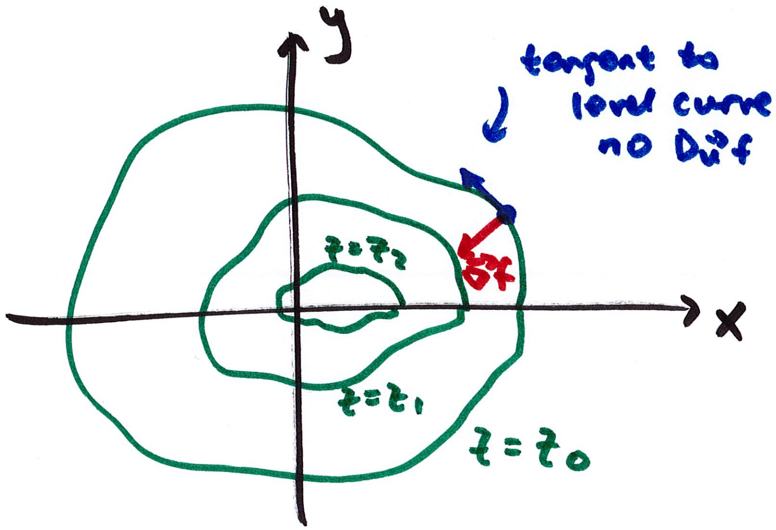
in other words, the direction of the gradient is the direction of maximum ~~derivative~~ directional ~~direc~~ derivative

the direction of gradient is also called the direction of steepest ascent

$D_{\vec{u}} f$ is a minimum when $\vec{\nabla} f$ and $\vec{u}$ are in opposite directions (steepest descent)

$D_{\vec{u}} f = 0$ if $\vec{\nabla} f \perp \vec{u}$

what is the significance of this direction?

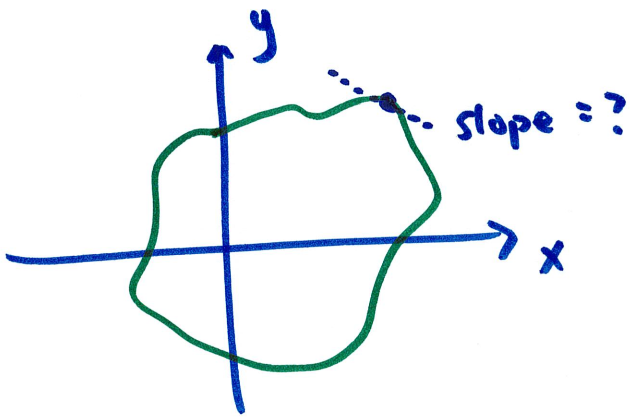


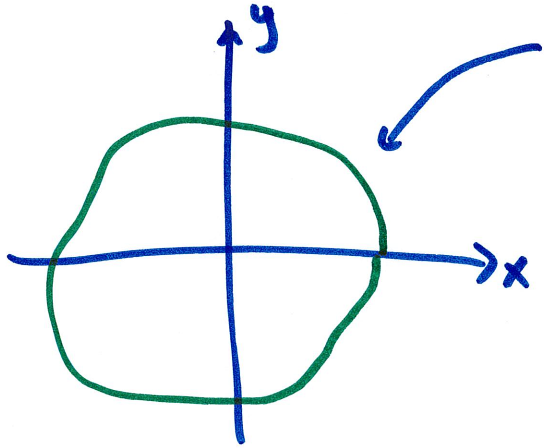
traveling on level curve → z does not change → $D_{\vec{u}} f = 0$ in that direction

⇒ travel <u>tangent</u> to level curve

$\boxed{\text{so, the gradient must be orthogonal to a level curve}}$

slope of level curve at a point?





level curve is a space curve

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

then at (x,y) we know

$$D_{\vec{u}} f(x,y) = \vec{\nabla} f(x,y) \cdot \vec{u}$$

w/ along level curve: $\vec{u} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\langle x'(t), y'(t) \rangle}{|\langle x'(t), y'(t) \rangle|}$

$$D_{\vec{u}} f = 0 = \langle f_x, f_y \rangle \cdot \frac{\langle \frac{dx}{dt}, \frac{dy}{dt} \rangle}{|\langle \frac{dx}{dt}, \frac{dy}{dt} \rangle|} = 0$$

so, $f_x \cdot \dfrac{dx}{dt} + f_y \cdot \dfrac{dy}{dt} = 0$

or $\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\dfrac{f_x}{f_y} \rightarrow \boxed{\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}}$

is the same as $\dfrac{dy}{dt} \cdot \dfrac{dt}{dx} = \dfrac{dy}{dx}$

slope of the level curve at a point

example

$f(x,y) = xe^y$

slope of level curve at $(1, 2)$?

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{e^y}{xe^y} = -\frac{1}{x} \qquad \text{at } (1,2) \quad \frac{dy}{dx} = -1$$

we also notice that whenever $x=0$, $\frac{dy}{dx}$ is undefined

so the level curve is vertical whenever $x=0$