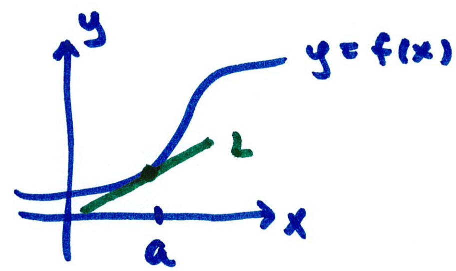
15.6 Tangent Plane and Linear Approximation

(NOT on exam 1)

recall if $y = f(x)$

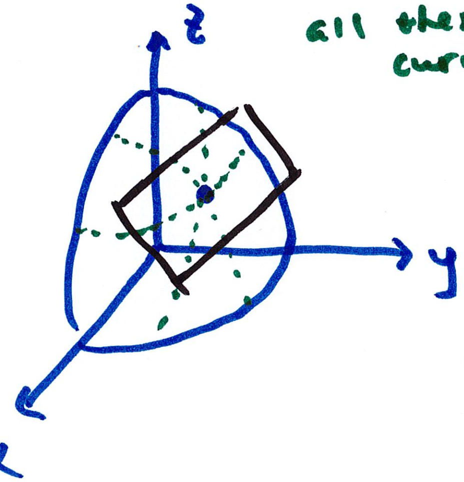


near $x = a$
the tangent line L is approximately equal to the true curve

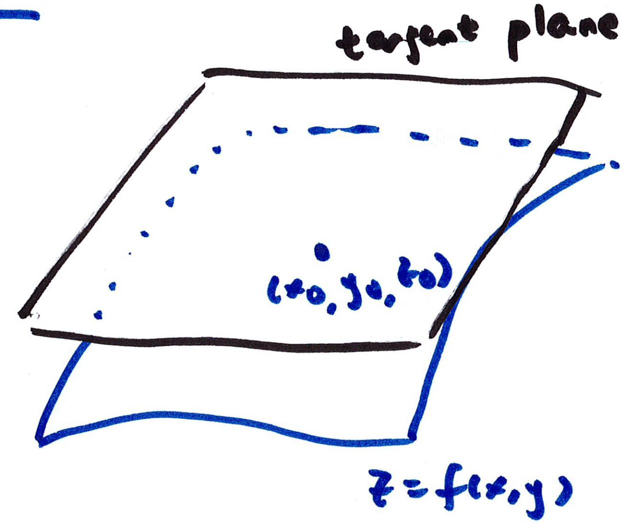
$$f(x) \approx L = f(a) + f'(a)(x-a)$$

the closer x is to a, the better the approximation

$z = f(x,y)$ is a surface and at point (x_0, y_0, z_0) there are infinitely-many tangent lines and the form a tangent plane



all these curves are part of the surface and each has a tangent line
they form a tangent plane



how to find equation of this tangent plane?

the surface is made up of a bunch of curves : $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

~~there~~ their tangent vectors $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$ form the tangent plane

the normal vector of the tangent plane must be normal to all $\vec{r}\,'(t)$

what is that vector?

$z = f(x, y)$ is the surface

it's made up of ~~all~~ a bunch of $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

so, $z(t) = f(x(t), y(t))$

let $F(x,y,z) = f(x(t), y(t)) - z(t) = 0 = G(t)$ → what F looks like as function of t

$$\begin{tikzcd}
 & F \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
x \arrow[d, dash] & y \arrow[d, dash] & z \arrow[d, dash] \\
t & t & t
\end{tikzcd}$$

by Chain Rule, $\dfrac{\partial F}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial F}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial F}{\partial z}\dfrac{dz}{dt} = 0$ ← because $F = f - z = 0$

$$\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = 0$$

$\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle = \vec{r}\,'(t)$ tangent vectors which make the tangent plane

and we see that $\vec{\nabla} F = \langle F_x, F_y, F_z \rangle$ is normal to it

so, the equation of tangent plane at (x_0, y_0, z_0) is

$$\boxed{\begin{gathered}\langle F_x, F_y, F_z \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0 \\ \text{or } F_x(x-x_0) + F_y(y-y_0) + F_z(z-z_0) = 0\end{gathered}}$$

where $F(x,y,z) = f(x,y) - z$

if we prefer using $z = f(x,y)$ explicitly, then adjust the above

$F = f - z \rightarrow F_x = f_x, \; F_y = f_y, \; F_z = -1$

$$\boxed{\begin{gathered}f_x(x-x_0) + f_y(y-y_0) - (z-z_0) = 0 \\ \text{or } \quad z - z_0 = f_x(x-x_0) + f_y(y-y_0)\end{gathered}}$$

example $z = f(x,y) = \sqrt{x^2+y^2}$

find tangent plane at $(3,4,5)$

let $F = f - z = \sqrt{x^2+y^2} - z$

then $\vec{\nabla} F = \langle F_x, F_y, F_z \rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle$

$\vec{\nabla} F(3,4,5) = \left\langle \frac{3}{5}, \frac{4}{5}, -1 \right\rangle$

tangent line equation: $\left\langle \frac{3}{5}, \frac{4}{5}, -1 \right\rangle \cdot \langle x-3, y-4, z-5 \rangle = 0$

$$\boxed{\frac{3}{5}(x-3) + \frac{4}{5}(y-4) - (z-5) = 0}$$

$\downarrow$

near $(3,4,5)$, $z = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4) \approx f(x,y) = \sqrt{x^2+y^2}$

how good is the approximation at $x = 3.01$, $y = 3.99$?

tangent plane/linear approx: $z = 5 + \frac{3}{5}(3.01-3) + \frac{4}{5}(3.99-4) = 4.998$

true z: $f(x,y) = \sqrt{x^2+y^2} = \sqrt{(3.01)^2 + (3.99)^2} = 4.99802$ not bad look at the tangent plane eq $z - z_0 = f_x(x - x_0) + f_y(y - y_0)$ again

close to (x_0, y_0, z_0), $x - x_0 = dx$ (very small change in x)

$$y - y_0 = dy$$

$$z - z_0 = dz$$

the tangent plane becomes $dz = f_x\,dx + f_y\,dy$

this is often used to estimate change in z (or error) given changes in x and y (or errors)

<u>example</u> $z = f(x,y) = x^2 y$

if x starts at 1 and increases by 0.01

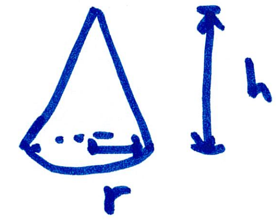
if y " " 3 " decreases " 0.09

by how much does z approximately change?

$$dz = f_x\,dx + f_y\,dy$$

$$= (2xy)\underbrace{dx}_{=\,0.01} + (x^2)\underbrace{dy}_{=\,-0.09} = (2\cdot 1\cdot 3)(0.01) + (1^2)(-0.09) = -0.03$$

x, y refer to the "old" or starting values of x, y because that's where the tangent plane is built example The volume of a cone is $V = \frac{1}{3}\pi r^2 h$



if r is increased by 1 %

and h is decreased by 3 %

what is the approximate % change in volume ?

$$V = \frac{1}{3}\pi r^2 h$$

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh$$

$$\underline{\underline{dV}} = \left(\frac{2}{3}\pi r h\right)dr + \left(\frac{1}{3}\pi r^2\right)dh$$

↳ change in V NOT % change in V

divide by $V = \frac{1}{3}\pi r^2 h$

% change in V

$$\frac{dV}{V} = \frac{\frac{2}{3}\pi r h}{\frac{1}{3}\pi r^2 h}dr + \frac{\frac{1}{3}\pi r^2}{\frac{1}{3}\pi r^2 h}dh$$

$$\frac{dV}{V} = 2\underbrace{\frac{dr}{r}}_{\text{% change in r}} + \underbrace{\frac{dh}{h}}_{\text{% change in h}} = 2\underbrace{(0.01)}_{\text{1% increase}} + \underbrace{(-0.03)}_{\text{3% decrease}} = -0.01 = 1\%\ \text{decrease}$$