15.7 Maximum and Minimum Problems (part 1)

recall if $y = f(x)$ then $f'(c) = 0 \rightarrow$ there is a critical number $x = c$

critical point $(c, f(c))$

the **Second Derivative Test** tells us that

$\cup$ if $f''(c) > 0 \rightarrow$ ~~max~~ relative min at $(c, f(c))$

$\cap$ if $f''(c) < 0 \rightarrow$ relative max at $(c, f(c))$

if $f''(c) = 0 \rightarrow$ test is inconclusive

for $z = f(x, y)$, the Second Derivative Test looks a bit different, but the basic idea is the same – treat the surface as if it were a paraboloid-like shape near critical points

Critical point: (a, b) where both $f_x(a,b)$ and $f_y(a,b)$ are zero then we form the discriminant $D = f_{xx} f_{yy} - (f_{xy})^2$

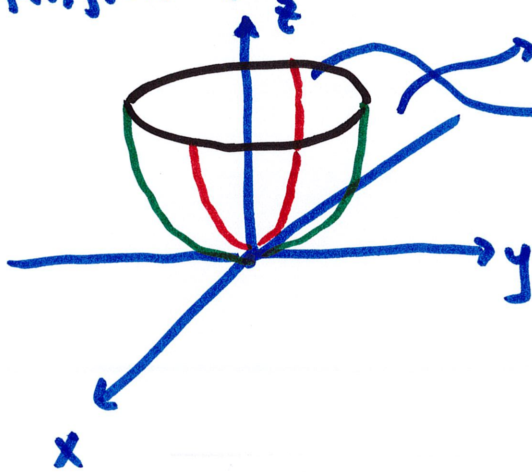
if at critical point (a,b) $f_{xx} > 0$ and $D > 0$ then there is a relative minimum at (a,b)

if at critical point (a,b) $f_{xx} < 0$ and $D > 0$ there there is a relative maximum

if at critical point (a,b) $D < 0$ then there is a saddle point

if at critical point (a,b) $D = 0$ then the test is inconclusive this D tells us a bit about the shape near (a,b)

for example, $f(x,y)=x^2+y^2$



green: $f_{yy}>0$ (concave up)

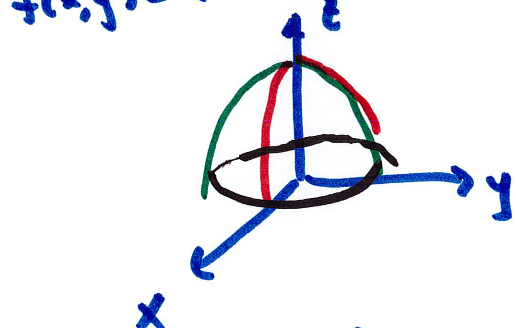
red: $f_{xx}>0$ (concave up)

$D>0 \rightarrow f_{xx}, f_{yy}$ same sign

(that's why we only need to look at f_{xx})

$D>0,\ f_{xx}>0 \rightarrow$ min at critical point

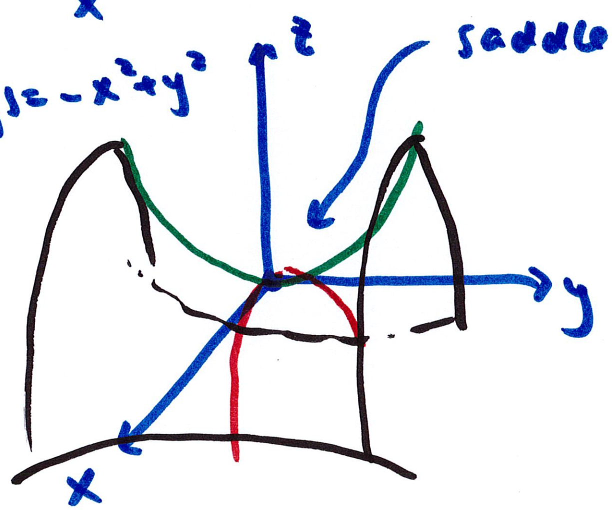
$f(x,y)=4-x^2-y^2$



here, $f_{xx}<0$, $f_{yy}<0$, $D>0$

max at critical point

$f(x,y)=-x^2+y^2$



saddle point

$$\left.\begin{array}{l} f_{yy}>0 \\ f_{xx}<0 \end{array}\right\} D<0$$

saddle point $\rightarrow$ neither max nor min example $f(x,y) = x^3 - 48xy + 64y^3$

Critical points: $f_x = 0$ and $f_y = 0$

$f_x = 3x^2 - 48y = 0$ - 1

$f_y = -48x + 192y^2 = 0$ - 2

from 1, $x^2 = 16y$ - 3

from 2, $x = 4y^2$ - 4

sub 4 into 3 $(4y^2)^2 = 16y$

$16y^4 = 16y$

$16y^4 - 16y = 0$

$16y(y^3 - 1) = 0 \rightarrow y = 0, \ y = 1$

critical point has x and y, so we need corresponding x

from 4, $y = 0 \rightarrow x = 0$
$y = 1 \rightarrow x = 4$

critical points: $(0,0)$, $(4,1)$

now form $D = f_{xx} f_{yy} - (f_{xy})^2$

$f_{xx} = 6x, \quad f_{yy} = 384y, \quad f_{xy} = -48 = f_{yx}$

$D = 2304xy - 2304$

at $(0,0)$, $D < 0 \rightarrow$ $\boxed{\text{saddle point}}$

at $(4,1)$ $D > 0$, $f_{xx} = 6x > 0 \rightarrow$ $\boxed{\text{rel. min at } (4,1)}$

example $f(x,y) = xye^{-x^2-y^2}$

find critical points: $f_x = 0$ AND $f_y = 0$

$$f_x = \cdots = ye^{-x^2-y^2}(-2x^2+1) = 0 \quad - (1)$$

$$f_y = \cdots = xe^{-x^2-y^2}(-2y^2+1) = 0 \quad - (2)$$

$$\underbrace{\left(e^{-x^2-y^2}\right)}_{\text{exponential NEVER zero}}(y)(-2x^2+1) = 0$$

so, $y = 0$ or $-2x^2+1 = 0 \rightarrow x = \pm\frac{1}{\sqrt{2}}$

DO NOT pair these up to form a point because these make $f_x = 0$ but are not guaranteed to make $f_y = 0$ and critical points MUST be where BOTH f_x and f_y are zero from ① : $y=0$, $x=\pm\frac{1}{\sqrt{2}}$

from ② : $\underbrace{(e^{-x^2-y^2})}_{\neq 0}(x)(2y^2-1)=0 \rightarrow x=0$, $2y^2-1=0$

$y=\pm\frac{1}{\sqrt{2}}$

choose one from each to form critical pts

$(0,0)$, $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$

then that D thing $D = f_{xx}f_{yy}-(f_{xy})^2$

$f_{xx} = \cdots = 2xy(2x^2-3)e^{-x^2-y^2}$

$f_{yy} = \cdots = 2xy(2y^2-3)e^{-x^2-y^2}$

$f_{xy} = \cdots = (2x^2-1)(2y^2-1)e^{-x^2-y^2}$

at $(0,0)$, $D<0 \rightarrow$ saddle point

at $(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}) \rightarrow$ rel. max

rel. min at $(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$

what if $D = 0$?

<u>example</u> $f(x,y) = xy^2$

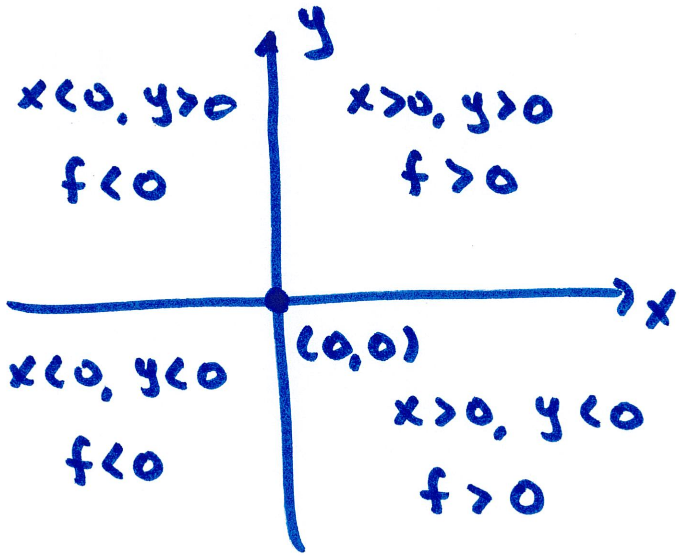
$\vdots$

$(0,0)$ is the only critical point

$f_x = y^2$, $f_y = 2xy$, $f_{xy} = 2y$, $f_{yy} = 2x$, $f_{xx} = 0$

$D = -4y^2 \rightarrow D = 0$ at $(0,0)$ test is inconclusive

let's examine the points near $(0,0)$ to determine its type



at $(0,0)$ $f = 0$

move right, $f > 0$

move left, $f < 0$

$\rightarrow$ so $(0,0)$ is <u>NOT</u> the location of a max or min