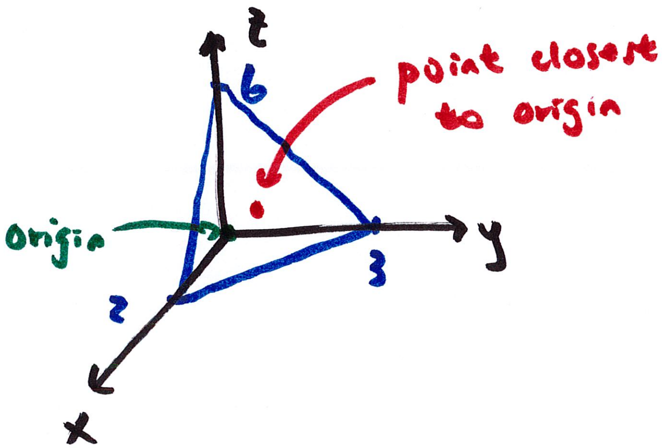
15.7 Max and Min Problems (part 2)

example What point on the plane $3x+2y+z=6$ is the closest to the origin?



points on the plane: (x, y, z)

from plane equation: $z = 6-3x-2y$

$(x, y, 6-3x-2y)$

distance from origin to any point on plane

$$d = \sqrt{(x-0)^2 + (y-0)^2 + (6-3x-2y-0)^2}$$

$$= \sqrt{x^2+y^2+(6-3x-2y)^2}$$

minimize this

need critical pts, second-order partials

square root makes derivative messy

since $d \geq 0$, we can minimize its square instead let $f = d^2 = x^2 + y^2 + (6-3x-2y)^2$ now partials are easier to find

$f_x = 2x + 2(6-3x-2y)(-3) = 0 \rightarrow 5x + 3y = 9$

$f_y = 2y + 2(6-3x-2y)(-2) = 0 \rightarrow 6x + 5y = 12$

solving them simultaneously, we get

$x = \frac{9}{7}, \; y = \frac{6}{7}$ one critical pt: $(9/7, 6/7)$

$$\left.\begin{array}{l} f_{xx} = 20 \\ f_{yy} = 10 \\ f_{xy} = 12 \end{array}\right\} \quad D = f_{xx} f_{yy} - (f_{xy})^2 = 200 - 144 > 0$$

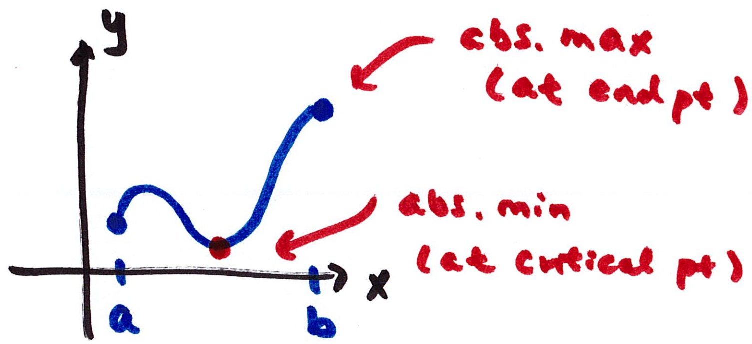
$D > 0, \; f_{xx} > 0 \rightarrow$ rel. min at critical pt

now we know the point on the plane closest to origin has

$x = 9/7, \; y = 6/7, \; z = 6 - 3x - 2y = 3/7$

So, the point is $\boxed{(9/7, 6/7, 3/7)}$

recall if $y=f(x)$, $a \le x \le b$, then the absolute max and min of $f(x)$ on $a \le x \le b$ can be at the critical pts inside $a \le x \le b$ or at the end points

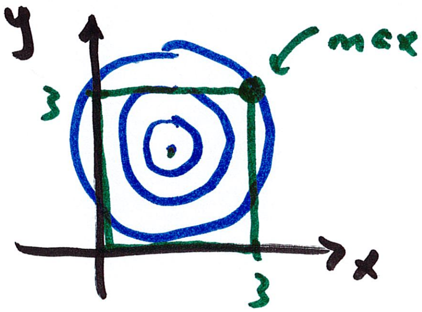


procedure: find interior cp
compare $y=f(x)$ at cp and end points to find max/min

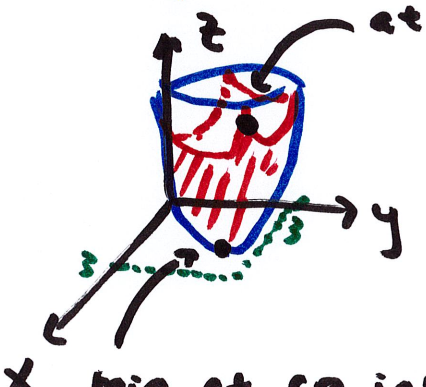
$z=f(x,y)$ is a surface, the domain restriction $\{(x,y)\}$ is a region on xy-plane, the boundary of it is a curve or set of curves on xy-plane

$z=f(x,y)=(x-1)^2+(y-2)^2$

over $0 \le x \le 3$ $0 \le y \le 3$



at boundary of region



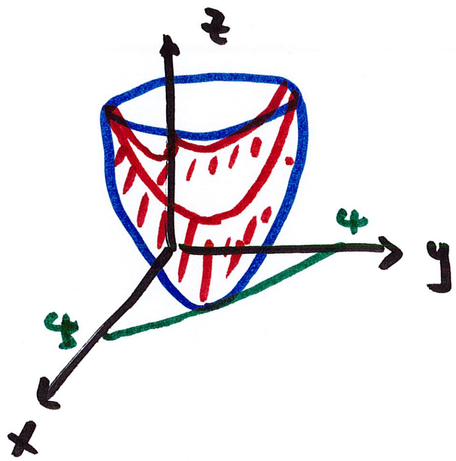
min at cp inside the region

we only care about the surface over the region $0 \le x \le 3$, $0 \le y \le 3$

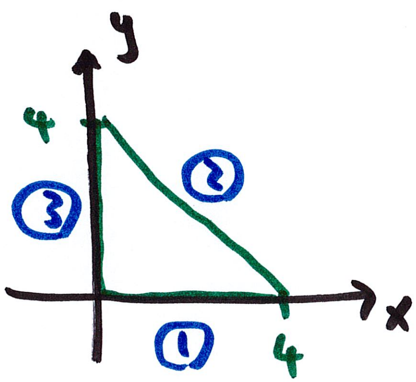
just like w/ $y = f(x)$, $a \le x \le b$, the max/min can occur at interior cp's and on the boundary of the region

<u>example</u> $z = f(x,y) = x^2 + y^2 - 2x - 4y + 10$

above the triangular region with vertices $(0,0)$, $(0,4)$, $(4,0)$



find highest/lowest pts on the red part of the paraboloid (which casts a triangular shadow)



the region ~~can be~~ has the following boundary curves:

① $y = 0$, $0 \le x \le 4$

② $y = 4 - x$, $0 \le x \le 4$

③ $x = 0$, $0 \le y \le 4$

find cp's of $f(x,y) = x^2+y^2-2x-4y+10$ inside the triangular shadow

$f_x = 2x-2 = 0$

$f_y = 2y-4 = 0$

$\boxed{\text{cp: } (1,2)}$

inside shadow?

yes, so we keep it

now we examine the boundary curves to find locations where max/min might be

① $y=0, \quad 0 \le x \le 4$

$f(x,y) = x^2+y^2-2x-4y+10$ becomes $f(x) = x^2-2x+10, \quad 0 \le x \le 4$

this is now a one-variable max/min problem

$f'(x) = 2x-2 = 0 \rightarrow x=1, \; y=0$

end pts: $x=0, \; y=0$

$x=4, \; y=0$

$\boxed{\text{points of interest: } (1,0), (0,0), (4,0)}$

$f(x,y) = x^2 + y^2 - 2x - 4y + 10$

② $y = 4 - x, \quad 0 \le x \le 4$

$f(x) = x^2 + (4-x)^2 - 2x - 4(4-x) + 10$

$f(x) = 2x^2 - 6x + 10, \quad 0 \le x \le 4$

$f' = 4x - 6 = 0 \rightarrow x = 3/2, \quad y = 4 - x = 5/2$

end pts: $x = 0, \; y = 4$

$x = 4, \; y = 0$

$$\boxed{\begin{array}{l} \text{pts of interest: } (3/2, 5/2) \\ (0, 4) \\ (4, 0) \end{array}}$$

repeat w/ ③ $x = 0, \quad 0 \le y \le 4$

$f(y) = y^2 - 4y + 10, \quad 0 \le y \le 4$

$f' = 2y - 4 = 0 \rightarrow y = 2, \; x = 0$

end pts: $y = 0, \; x = 0$

$y = 4, \; x = 0$

$$\boxed{\text{pts: } (0,2), (0,0), (0,4)}$$

now we compare $f(x,y) = x^2+y^2-2x-4y+10$ at the pts collected

cp: $(1,2)$, $(0,0)$, $(4,0)$, $(1,0)$, $(0,4)$, $(3/2, 5/2)$, $(0,2)$

$f(1,2) = 5$ $\rightarrow$ abs. min at $(1,2,5)$ (vertex)

$f(0,0) = 10$

$f(4,0) = 18$ $\rightarrow$ abs. max at $(4,0,18)$

$f(1,0) = 9$

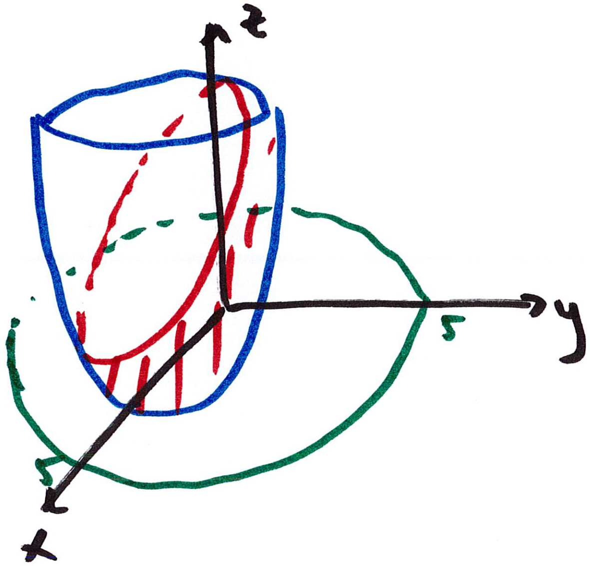
$f(0,4) = 10$

$f(3/2, 5/2) = 11/2$

$f(0,2) = 6$

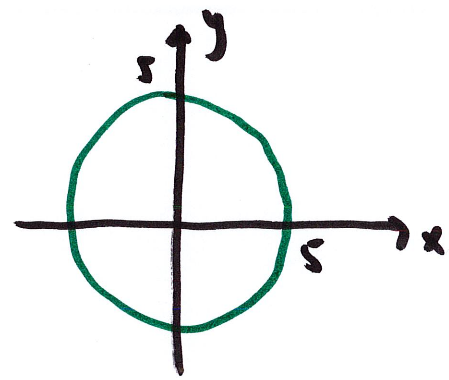
example Find abs. max/min of $f(x,y) = x^2+y^2-6x+9$

above the region $x^2+y^2 \le 25$



again, find highest/lowest of the red part

which casts a circular shadow (circle radius 5)



note we can't chop this into straight boundaries

it's ok: we know on the boundary $x^2+y^2=25$

and $-5 \le x \le 5$, $y = \pm\sqrt{25-x^2}$

or $-5 \le y \le 5$, $x = \pm\sqrt{25-y^2}$

$$f(x,y) = \underbrace{x^2+y^2}_{25} - 6x + 9$$

leave as x and use $-5 \le x \le 5$

$f(x) = 25 - 6x + 9 = 34 - 6x$

$f' = -6 \ne 0$ so no cp on boundary

end pts: $x=-5$, $y = \pm\sqrt{25-x^2} = 0$

$x=5$, $y=0$

pts of interest on boundary: $(-5,0)$, $(5,0)$

$f(x,y) = x^2+y^2-6x+9$

$f_x = 2x-6=0 \to x=3$

$f_y = 2y=0 \to y=0$

$\Big\}$ interior cp: $(3,0)$

compare $f(x,y) = x^2+y^2-6x+9$ at $(3,0)$, $(-5,0)$, $(5,0)$

$f(3,0) = 0$ abs. min

$f(-5,0) = 64$ abs. max

$f(5,0) = 4$