15.8 Lagrange Multipliers

constrained optimization: find max/min of something subject to some condition(s)

for example: find max/min of $f(x,y) = x^2+y^2$ subject to condition $xy=1$

x, y must obey $xy = 1$

$f(x,y) = x^2+y^2$ is called the objective (goal)

$g(x,y) = xy-1 = 0$ is called the constraint

if the functions are simple, we can just do substitutions

$f(x,y) = x^2+y^2$ $\quad xy=1 \rightarrow y = \frac{1}{x}$

$f(x) = x^2 + \left(\frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2}$

$f'(x) = 2x - \frac{2}{x^3} = 0 \rightarrow x^4 = 1 \rightarrow x = 1, -1$

then since $y = \frac{1}{x}$, the critical pts are $(1, 1)$, $(-1, -1)$

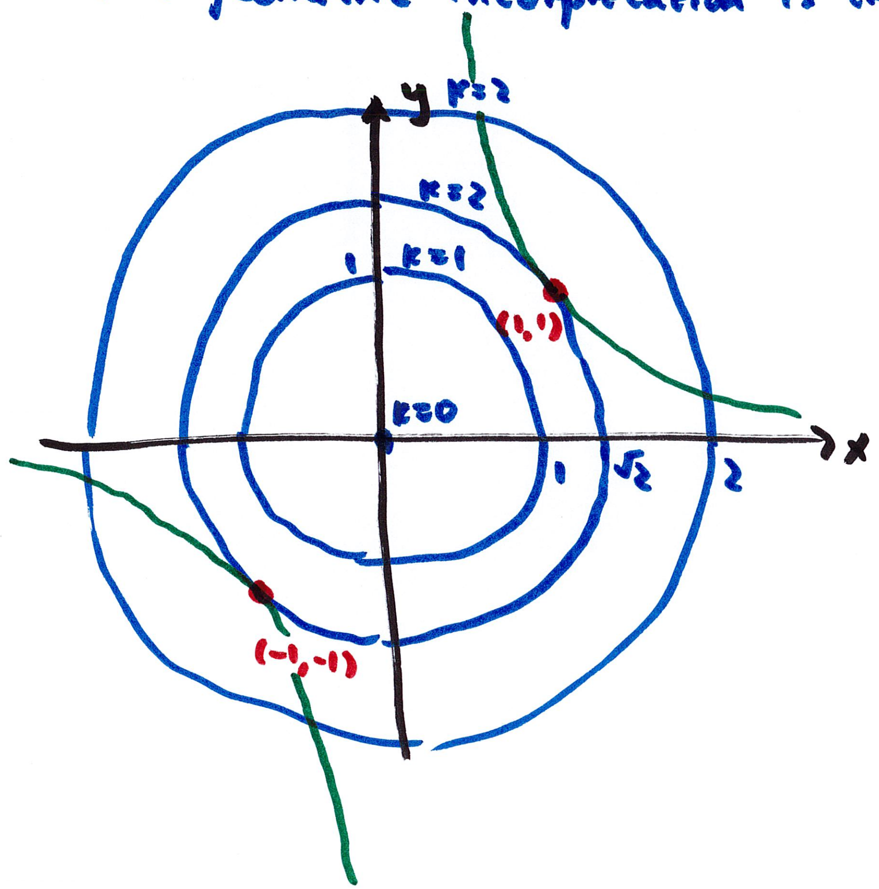
$f(x,y) = x^2+y^2$, $xy=1$

f does not have a max because we can make either x or y arbitrarily large (the other is $y=\frac{1}{x}$ or $x=\frac{1}{y}$)

so, the critical pts must be locations of minimum f

(min $f(x,y) = 1^2+1^2 = 2$)

the geometric interpretation is the most important thing here



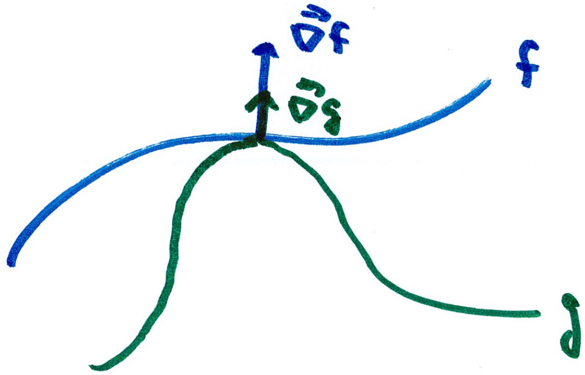
level curves of $f(x,y) = x^2+y^2 = k$

$xy=1 \rightarrow y=\frac{1}{x}$ hyperbola

notice at the locations of max/min f, the constraint barely touches the level curve

that's also where any additional movement on the constraint results in a higher value of f the point where f barely touches g → where they are <u>tangent</u>

if f and g are tangent, then their gradients must also point in the same direction



so, this means we look for max/min of constrained optimization problems at places where f and g are tangent

mathematically → $\vec{\nabla} f = \lambda \vec{\nabla} g$

↳ constant called Lagrange Multiplier (Greek lambda)

this is the basis of the method of Lagrange Multipliers

1. solve for (x, y) where $\vec{\nabla} f = \lambda \vec{\nabla} g$
2. Compare f at those places

example $f(x,y) = 4 - x^2 - y^2$ subject to the constraint $\underbrace{4x^2 + y^2 = 4}$

$$\downarrow$$

$g(x,y) = 4x^2 + y^2 - 4 = 0$

$\vec{\nabla} f = \langle -2x, -2y \rangle$

$\vec{\nabla} g = \langle 8x, 2y \rangle$

solve $\vec{\nabla} f = \lambda \vec{\nabla} g$ for x, y, λ

$\langle -2x, -2y \rangle = \lambda \langle 8x, 2y \rangle$

$-2x = \lambda \cdot 8x$ — ①

$-2y = \lambda \cdot 2y$ — ②

① : $\lambda \cdot 8x + 2x = 0$

$2x(4\lambda + 1) = 0 \rightarrow 2x = 0$ or $4\lambda + 1 = 0$

$x = 0$ or $\boxed{\lambda = -1/4}$

put into $g(x,y)$ to find y : $y^2 - 4 = 0$

$\boxed{(0, 2), (0, -2)}$

from ②: $\lambda \cdot 2y + 2y = 0$

$2y(\lambda+1) = 0 \rightarrow 2y = 0$ or $\lambda + 1 = 0$

$y = 0$ or $\lambda = -1$

↳ put into $g(x,y)$ to find x: $4x^2 - 4 = 0$

$x = \pm 1$

$$\boxed{(1,0),\ (-1,0)}$$

from ①, $\lambda = -1/4$

if sub into ②, we get $-2y = (-1/4) \cdot 2y$

(same thing results if we sub $\lambda = -1$ into ①)

$-2y = -\frac{1}{2}y \rightarrow y = 0$

↳ same thing solving ② got us, so no new info.

now we compare $f(x,y) = 4 - x^2 - y^2$ at the points of interest

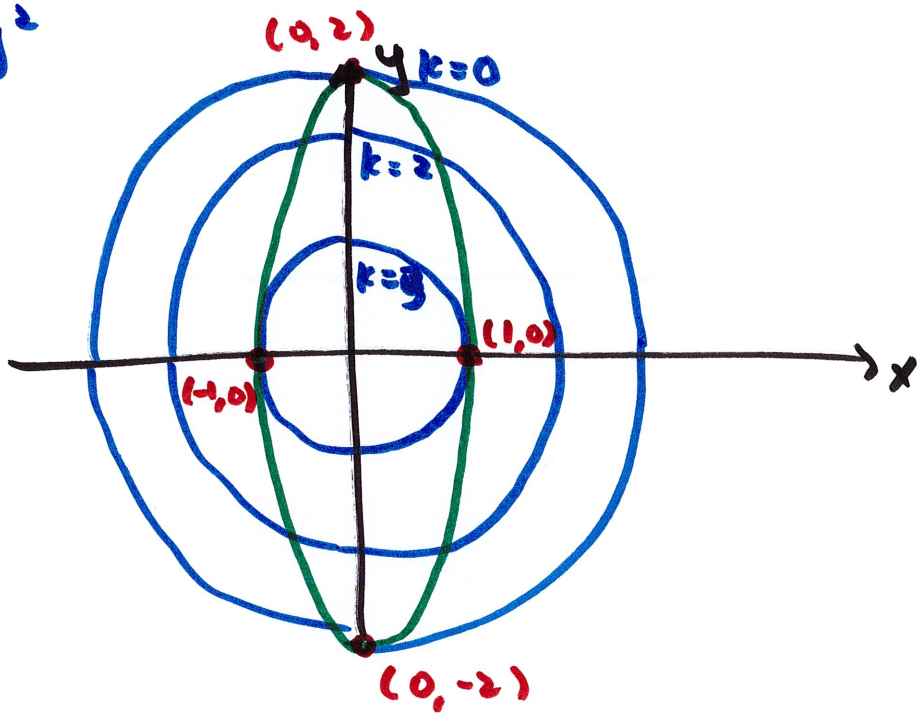
from ① $\left\{ \begin{array}{l} f(0,2) = 0 \\ f(0,-2) = 0 \end{array} \right\}$ 2 minima of 0 at $(0, \pm 2)$

from ② $\left\{ \begin{array}{l} f(1,0) = 3 \\ f(-1,0) = 3 \end{array} \right\}$ 2 maxima of 3 at $(\pm 1, 0)$

$f(x,y) = \underbrace{4 - x^2 - y^2}_{\text{circular level curves}}$

$k = 4 - x^2 - y^2$

$g(x,y) = 4x^2 + y^2 - 4 = 0 \rightarrow \underbrace{4x^2 + y^2 = 4}_{\text{ellipse}}$



example

$f(x,y,z) = xyz$

$g(x,y,z) = x^2+y^2+z^2-3 = 0$

interpretation: stay on sphere $x^2+y^2+z^2=3$ and find where the product of coordinates is max/min

solve $\vec{\nabla} f = \lambda \vec{\nabla} g$

$\langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$

$$\left.\begin{array}{lll} yz = \lambda \cdot 2x & - ① & \rightarrow \lambda = \frac{yz}{2x} \\ xz = \lambda \cdot 2y & - ② & \rightarrow \lambda = \frac{xz}{2y} \\ xy = \lambda \cdot 2z & - ③ & \rightarrow \lambda = \frac{xy}{2z} \end{array}\right\} \text{imply } \frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z}$$

$$\frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z}$$

$$\underbrace{y^2 z = x^2 z}_{x^2 = y^2} \qquad \underbrace{xz^2 = xy^2}_{z^2 = y^2}$$

$$\boxed{x^2 = y^2 = z^2}$$

↓

implies x, y, z can have different signs

sub into $g(x,y) = x^2 + y^2 + z^2 - 3 = 0$

$$x^2 + x^2 + x^2 = 3$$

$$x^2 = 1 \rightarrow x = \pm 1$$

same w/ y and z

$$y^2 = 1 \rightarrow y = \pm 1$$

$$z^2 = 1 \rightarrow z = \pm 1$$

8 critical pts

(1, 1, 1)

(1, −1, −1), (−1, 1, −1), (−1, −1, 1)

(1, 1, −1), (1, −1, 1), (−1, 1, 1)

(−1, −1, −1)

plug into $f(x,y,z) = xyz$ to find max/min