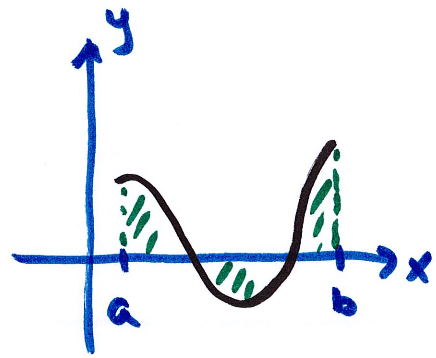
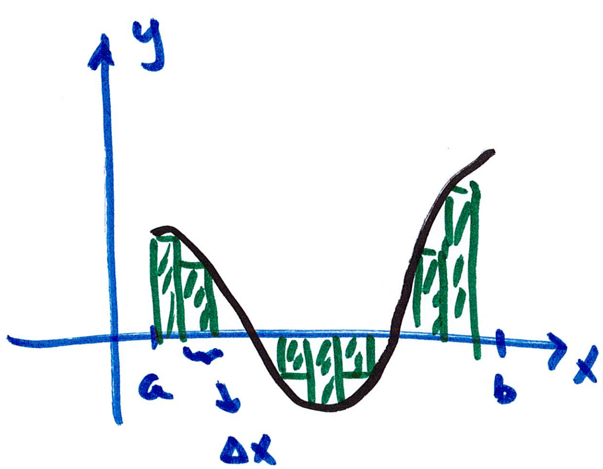
16.1 Double Integrals over Rectangular Regions

if $y = f(x)$, $a \leq x \leq b$



then $\int_a^b f(x)\,dx$ gives us the net area bounded by $f(x)$ and x-axis.

remember $\int_a^b f(x)\,dx$ is summing an infinite number of rectangles between $f(x)$ and x-axis



each rectangle has area $f(x_i)\Delta x$

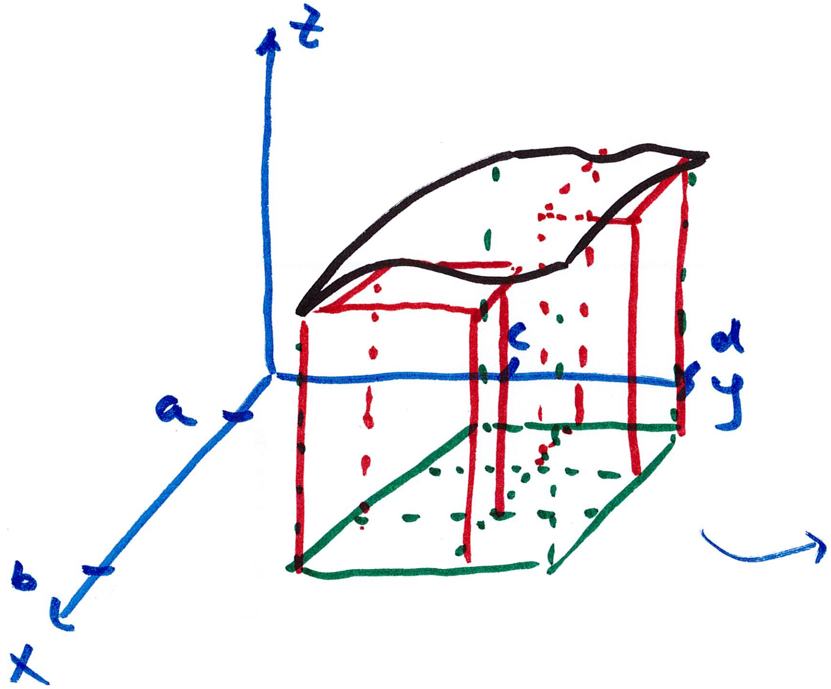
sample point

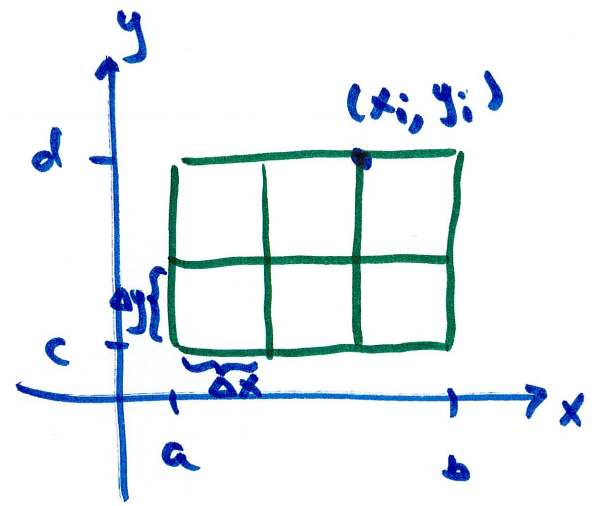
then as $n \to \infty$, the sum of n rectangles is

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$$

if we want volume of solid under $z = f(x,y)$ above the rectangle

$a \le x \le b$, $c \le y \le d$ $([a,b] \times [c,d])$, we use the same basic idea





each box has base area

$(\Delta x)(\Delta y) = \Delta A$

and height $\underbrace{f(x_i, y_j)}_{\text{sample point}}$

and volume

$f(x_i, y_j)\Delta A$

now we shrink Δx, Δy so $\Delta A \to dA$ and $f(x_i, y_j) \to f(x, y)$

$$\boxed{\text{total volume} = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\,\Delta A = \int_a^b \int_c^d f(x,y)\,dA}$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

y bounds: c, d

dA :

$dA = dx\,dy = dy\,dx$

or

$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$

x bounds

y bounds

Since base area is rectangle, the order doesn't matter (NOT the case if base is not rectangle) how to evaluate ~~dab~~ double integrals?

<u>example</u>

$$\int_0^1 \underbrace{\int_0^2 (3-x-y)\,dy}\,dx$$

basic idea: work inside-out

this inner integral is with respect to y → treat x as constant

$$= \int_0^1 \left(3y - xy - \frac{1}{2}y^2 \Big|_{y=0}^{y=2} \right) dx$$

$$= \int_0^1 (6 - 2x - 2 - 0)\,dx = \int_0^1 (4-2x)\,dx$$

$$= 4x - x^2 \Big|_0^1 = \boxed{3}$$

base is rectangle: $0 \le x \le 1$, $0 \le y \le 2$

so interchanging order doesn't change answer

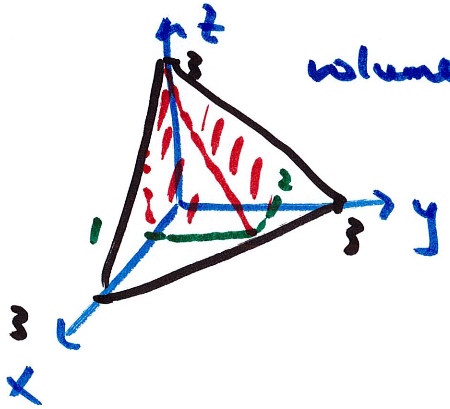
$$\int_0^1\int_0^2 (3-x-y)\,dy\,dx = \int_0^2 \underbrace{\int_0^1 (3-x-y)\,dx}_{y \text{ const}}\,dy$$

$$= \int_0^2 \left(3x - \frac{x^2}{2} - yx \Big|_{x=0}^{x=1}\right) dy$$

$$= \int_0^2 \left(3 - \frac{1}{2} - y\right) dy = \int_0^2 \left(\frac{5}{2} - y\right) dy$$

$$= \frac{5}{2}y - \frac{1}{2}y^2 \Big|_0^2 = 5 - 2 = \boxed{3}$$

physical interpretation: a volume under $z = 3-x-y$ above $0 \le x \le 1$, $0 \le y \le 2$



volume under portion that has rectangular shadow

Switching order can sometimes make integration easier (or harder)

<u>example</u>

$$\int_0^1 \underbrace{\int_0^2 y^5 x^2 e^{x^3y^3}\,dx}_{y \text{ is constant}}\,dy$$

$$\underbrace{y^5\int_0^2 x^2 e^{x^3y^3}\,dx}_{\text{Subs: } u = x^3y^3}$$

$$du = 3x^2y^3\,dx \rightarrow x^2dx = \cancel{\tfrac{1}{3}y^3\,du}\quad \frac{1}{3y^3}\,du$$

$$\cancel{y^5\int_{u=0}^{u=8y^3}\frac{1}{3}y^3 e^u\,du}$$

$$y^5\int_{x=0}^{x=2}\frac{1}{3y^3}e^u\,du = \frac{1}{3}y^2\int_{x=0}^{x=2}e^u\,du = \frac{1}{3}y^2 e^u\Big|_{x=0}^{x=2}$$

$$= \frac{1}{3}y^2 e^{x^3y^3}\Big|_{x=0}^{x=2} = \frac{1}{3}y^2 e^{8y^3} - \frac{1}{3}y^2$$

now finish the outer integral

$$\int_0^1 \left(\frac{1}{3}y^2 e^{8y^3} - \frac{1}{3}y^2\right) dy$$

$$= \underbrace{\int_0^1 \frac{1}{3}y^2 e^{8y^3}\, dy}_{\substack{u = 8y^3 \\ du = 24y^2\,dy}} - \underbrace{\int_0^1 \frac{1}{3}y^2\, dy}_{\text{easy}}$$

$$\vdots$$

$$= \boxed{\frac{1}{72}e^8 - \frac{1}{8}}$$

does the other order make the calculation easier?

$$\int_0^2 \underbrace{\int_0^1 y^5 x^2 e^{x^3y^3}\, dy}_{x^2 \int_0^1 y^5 e^{x^3y^3}\, dy}\, dx$$

maybe by parts, but probably not easier than original order example

$$\int_0^1 \underbrace{\int_0^{\pi/3} x^2 \cos(xy)\,dx}_{y \text{ is constant}}\,dy$$

by parts $u = x^2$ $dv = \cos(xy)\,dx$

~~ey~~ but needs Two rounds of by parts

examine the other order

$$\int_0^{\pi/3} \underbrace{\int_0^1 x^2 \cos(xy)\,dy}_{x \text{ is const.}}\,dx$$

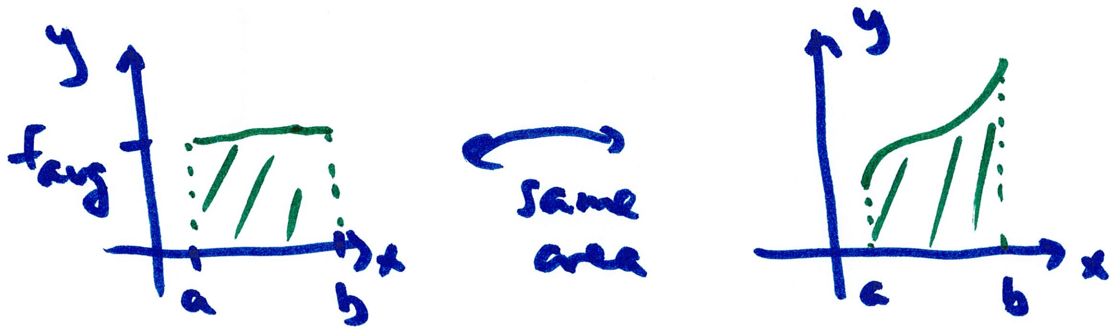
$$x^2 \cdot \sin(xy) \cdot \frac{1}{\cancel{x}} \Big|_{y=0}^{y=1} = x \sin(xy) \Big|_{y=0}^{y=1} = \dots$$

the outside requires one round of int. by parts recall if $y = f(x)$ $\quad a \le x \le b$

then the average value of $f(x)$ on $[a, b]$ is

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

this comes from the fact that a rectangle with base $b-a$ and height f_{avg} has the same area as $\int_a^b f(x)\,dx$



$$f_{avg} \cdot (b-a) \qquad\qquad \int_a^b f(x)\,dx$$

so $$f_{avg} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

using the same idea for $z = f(x,y)$ $\quad a \le x \le b$, $c \le y \le d$

the volume of box with base rectangle $[a,b] \times [c,d]$

and height f_{avg} has the same volume $\displaystyle\int_a^b \int_c^d f(x,y)\,dy\,dx$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$= \iint_R f(x,y)\,dA$$

$R: \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$

area of R: $(b-a)(d-c)$

so, $(f_{avg})(A) = \displaystyle\iint_R f(x,y)\,dA$

so, $$\boxed{f_{avg} = \frac{1}{A} \iint_R f(x,y)\,dA}$$