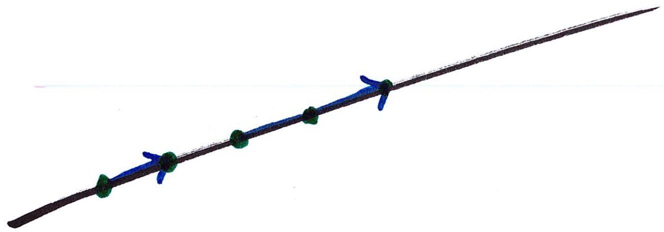
13.5 Lines and Planes

line



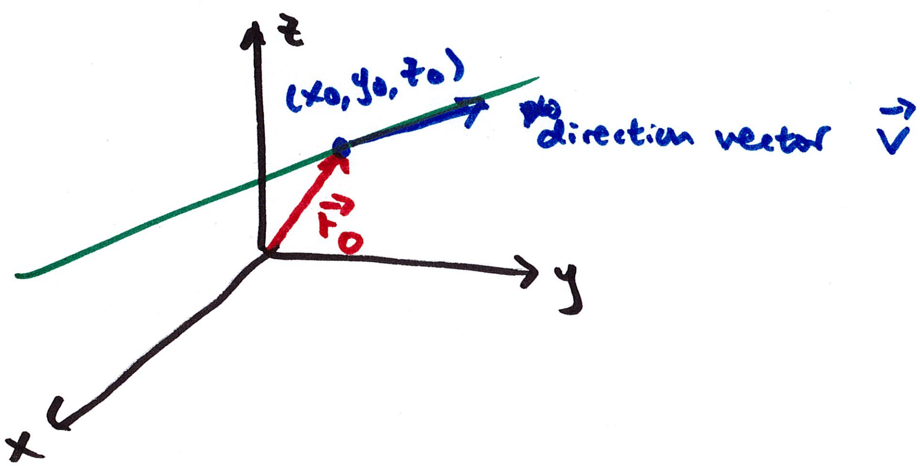
note the position vector from one point to another is always parallel to the position vector between another pair of points

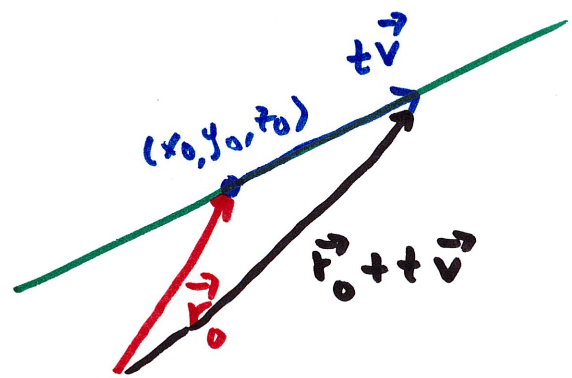
this vector is called the <u>direction vector</u> (like a slope)

if we know the position vector from the origin to any point on the line, then together with direction vector we can describe the line

let (x_0, y_0, z_0) be a point the line passes through

then $\langle x_0, y_0, z_0 \rangle$ is the position vector from origin : $\vec{r}_0$





by adjusting t, the tip of $\vec{r}_0 + t\vec{v}$ traces out every point on the line

the vector form of equation of line : $\boxed{\vec{r}(t) = \vec{r}_0 + t\vec{v}}$

let $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

$\vec{v} = \langle a, b, c \rangle$

then $\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$

$= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$

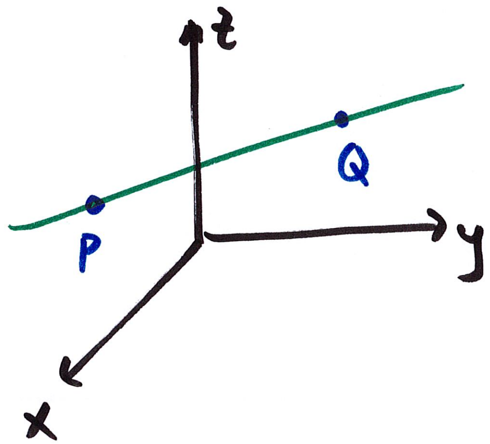
if we write out components

$$\boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}}$$

these form the parametric form of the equation of line

(t is the "parameter")

<u>example</u> Line through $P(0,1,2)$ and $Q(-3,4,7)$



direction vector : $\vec{v} = \vec{PQ}$ or $\vec{QP}$

let's do $\vec{PQ}$: $\vec{v} = \langle -3, 3, 5 \rangle$

then let P or Q be (x_0, y_0, z_0)

here, let P be (x_0, y_0, z_0)

position vector from origin to (x_0, y_0, z_0)

is $\vec{r}_0 = \langle 0, 1, 2 \rangle$

equation in vector form: $\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle 0, 1, 2 \rangle + t\langle -3, 3, 5 \rangle$

$-\infty < t < \infty$

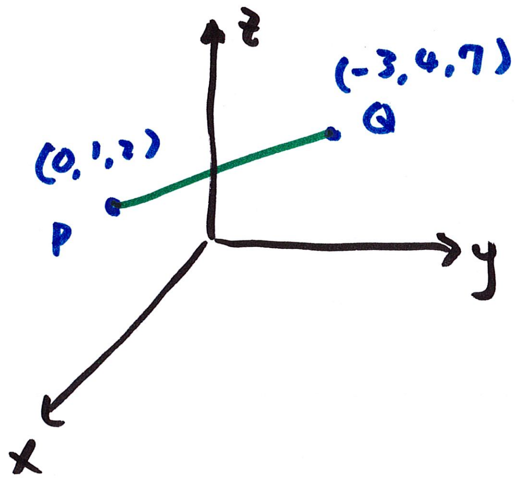
parametric form:

$x = -3t$

$y = 1+3t \qquad -\infty < t < \infty$

$z = 2+5t$

what if ~~I~~ we just want the segment between P and Q?



Start w/ the eq. for the infinite line

$\vec{r}(t) = \langle 0, 1, 2 \rangle + t \langle -3, 3, 5 \rangle$

note if $t=0$, $\vec{r}(0) = \langle 0, 1, 2 \rangle$ = position vector from origin to P

$t=0 \rightarrow$ point P

$t=? \rightarrow$ point Q $\quad (t=1)$

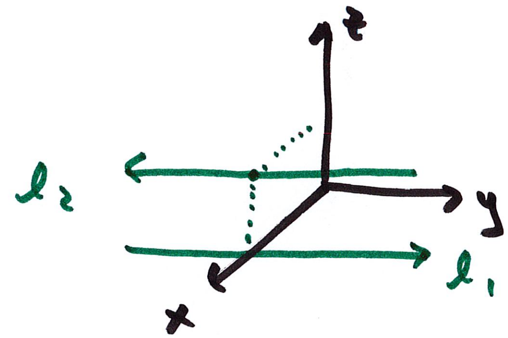
so, the eq. of segment is

$$\boxed{\begin{gathered}\vec{r}(t) = \langle 0, 1, 2 \rangle + t \langle -3, 3, 5 \rangle \\ 0 \le t \le 1\end{gathered}}$$

if the direction vectors of two lines are parallel, then the two lines are parallel

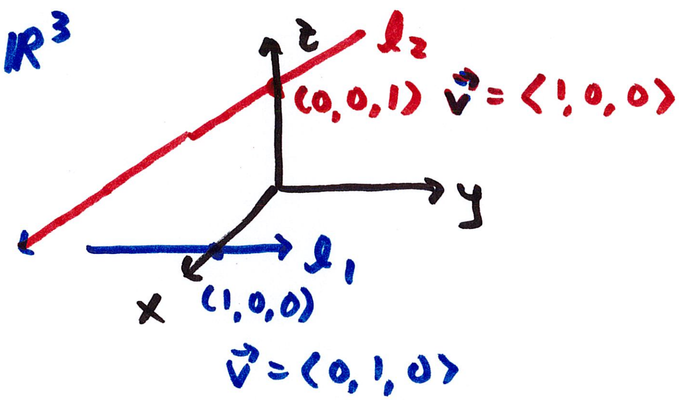
$\ell_1 \quad \vec{r}_1(t) = \langle 1, 0, 0 \rangle + t \langle 0, 1, 0 \rangle$

$\ell_2 \quad \vec{r}_2(t) = \langle 1, 0, 1 \rangle + t \langle 0, -2, 0 \rangle$



in $\mathbb{R}^2$, if two lines are not parallel, then they will intersect

not true in $\mathbb{R}^3$



these lines are neither ~~intersect~~ parallel nor will they intersect

→ they are <u>skew</u>

example Two objects travel on the lines

$$\vec{r}_1(t) = \langle 2t+3,\ 4t+2,\ 3t+5 \rangle \quad -\infty < t < \infty$$

$$\vec{r}_2(s) = \langle s+2,\ 3s-1,\ -5s+10 \rangle \quad -\infty < s < \infty$$

will the objects' paths intersect?

will the objects collide with each other?

intersect: can we find t, s such that $\vec{r}_1(t) = \vec{r}_2(s)$?

collide: can we find t, s such that $\vec{r}_1(t) = \vec{r}_2(s)$ AND $t = s$?

if $\vec{r}_1(t) = \vec{r}_2(s)$ then

x: $2t+3 = s+2$ — ①

y: $4t+2 = 3s-1$ — ②

z: $3t+5 = -5s+10$ — ③

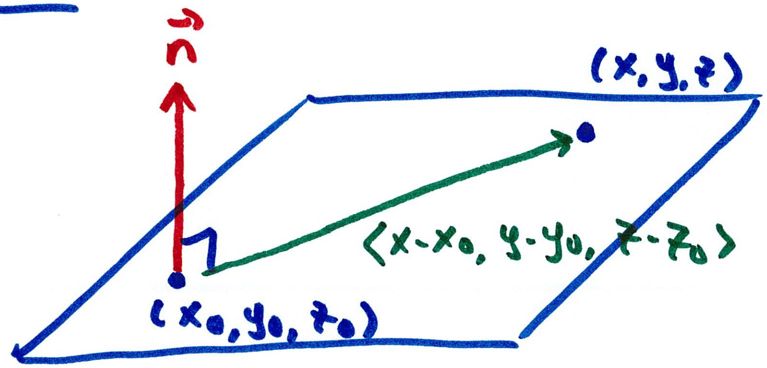
from ①, $s = 2t+1$

sub into ② $4t+2 = 3(2t+1)-1 \rightarrow \boxed{t=0 \quad \text{so,} \quad s=1}$ Check if these work in ③

③: $3(0)+5 = -5(1)+10$?

yes, so at $t=0$, $s=1$, $\vec{r}_1(t) = \vec{r}_2(s)$ they intersect but since $t \neq s$, the objects do NOT collide.

Plane



(x_0, y_0, z_0) : one point the plane passes through

(x, y, z) : some other point on the plane

$\vec{n}$: vector perpendicular (orthogonal or normal) to the plane

let $\vec{n} = \langle a, b, c \rangle$

then $\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$

$$\boxed{\begin{aligned} &= \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0 \\ &\text{or } a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \end{aligned}}$$

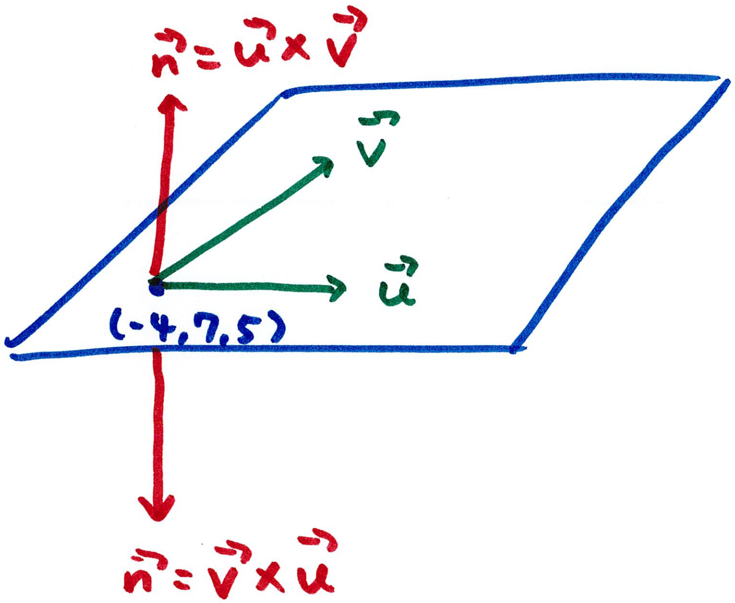
→ scalar

$\langle a, b, c \rangle$: normal vector

(x_0, y_0, z_0) : a point

example equation of plane through $\overset{x_0}{(-4},\ \overset{y_0}{7},\ \overset{z_0}{5)}$

and containing $\vec{u} = \langle 0, 1, 2 \rangle$ and $\vec{v} = \langle -1, -3, 0 \rangle$



normal vector

$\vec{n} = \vec{u} \times \vec{v}$ or $\vec{v} \times \vec{u}$

here, let $\vec{n} = \vec{u} \times \vec{v} = \langle 0, 1, 2 \rangle \times \langle -1, -3, 0 \rangle$

$$= \langle \underset{a}{+6}, \underset{b}{-2}, \underset{c}{+1} \rangle$$

$$\boxed{\begin{gathered} \text{equation: } +6(x+4) - 2(y-7) + (z-5) = 0 \\ \text{or} \\ +6x - 2y + z = 33 \end{gathered}}$$