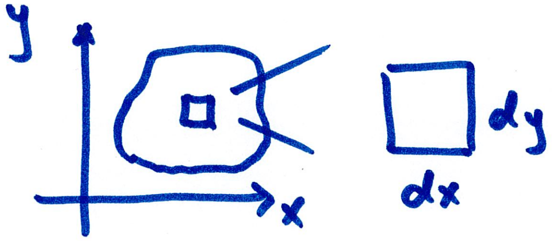
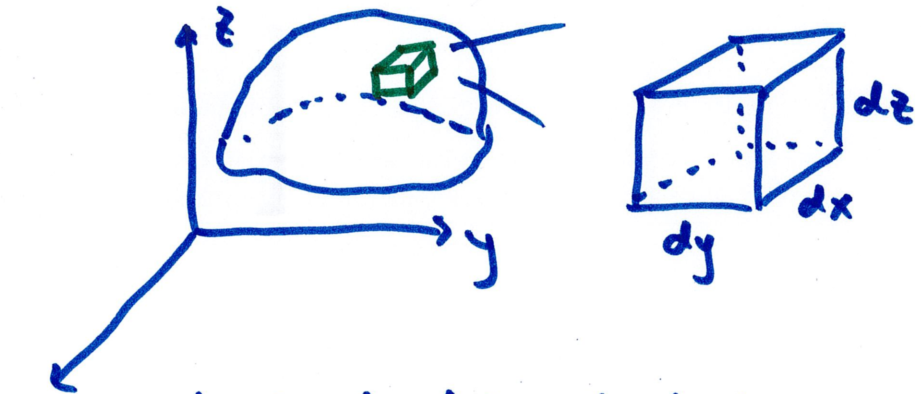
16.4 Triple Integrals

$\iint_R f(x,y)\,dA$ is the accumulation of $f(x,y)$ all over the region R



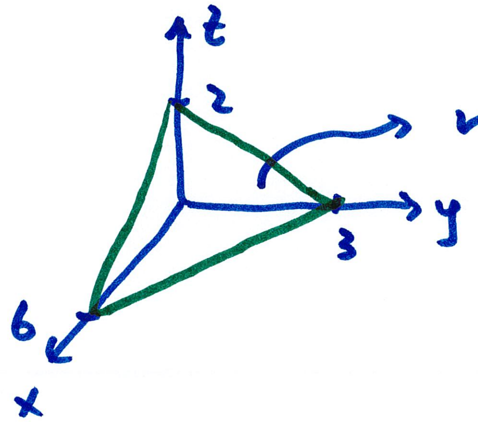
$dA = dy\,dx = dx\,dy$ two possible orders

$\iiint_D f(x,y,z)\,dV$ accumulates $f(x,y,z)$ all over the volume D



$$
\left.
\begin{aligned}
\text{volume of } dV &= dx\,dy\,dz = dx\,dz\,dy \\
&= dy\,dx\,dz = dy\,dz\,dx \\
&= dz\,dx\,dy = dz\,dy\,dx
\end{aligned}
\right\} \text{six possible orders}
$$

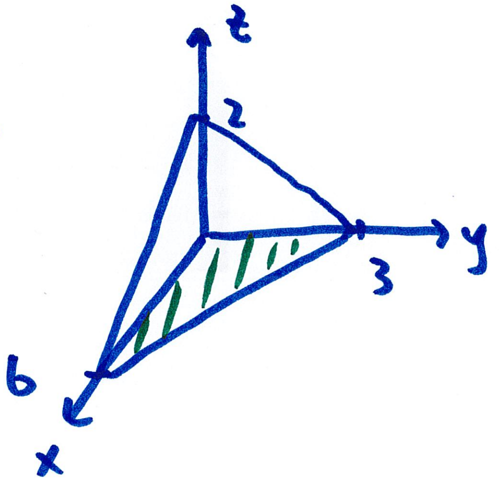
<u>example</u> Use a triple integral to calculate the volume of the solid under $x+2y+3z=6$ in the first octant.



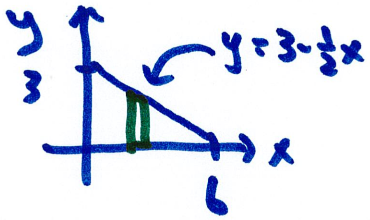
volume of this prism?

$$\text{volume} = \iiint_D dV$$

(just like $\iint_R dA$ is area)



look at the projection onto ~~[illegible]~~ xy-plane



Type I : $0 \le x \le 6$

$0 \le y \le 3 - \frac{1}{2}x$

(think of this as the "floor" of the prism)

now we ~~integrate~~ can go as high as the plane allows us to go ("ceiling")

$$\underset{\text{xy-plane}}{0} \le z \le \underbrace{2 - \frac{1}{3}x - \frac{2}{3}y}_{\text{from } x+2y+3z=6}$$

bounds: $0 \le x \le 6$

$0 \le y \le 3 - \frac{1}{2}x$

$0 \le z \le 2 - \frac{1}{3}x - \frac{2}{3}y$

Basic Rule: variable with constant bounds goes LAST (outside)

variable with bounds that include the most number of other variables goes FIRST (inside)

(or: simplest one is LAST, most complicated is FIRST)

here, x is outside, z is inside, y is middle

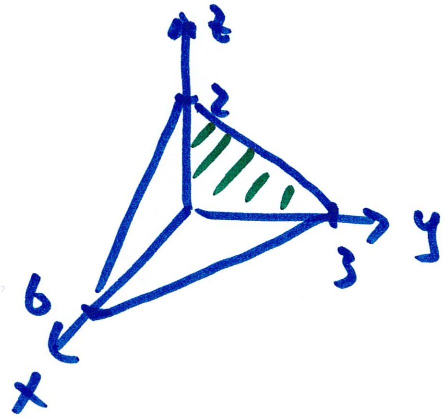
volume of prism: $$\int_0^6 \int_0^{3-\frac{1}{2}x} \int_0^{2-\frac{1}{3}x-\frac{2}{3}y} \overbrace{dz\,dy\,dx}^{dV}$$

x y z

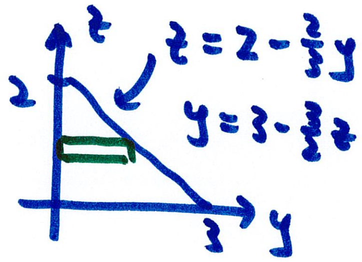
$$= \int_0^6 \int_0^{3-\frac{1}{2}x} z \Big|_{z=0}^{z=2-\frac{1}{3}x-\frac{2}{3}y} dy\,dx = \int_0^6 \int_0^{3-\frac{1}{2}x} \left(2 - \frac{1}{3}x - \frac{2}{3}y\right) dy\,dx$$

$$= \dots = \boxed{6}$$

let's try a different order with a different "floor"



use yz-plane as "floor"



Type II : $\boxed{0 \le z \le 2}$
$\boxed{0 \le y \le 3 - \frac{3}{2}z}$

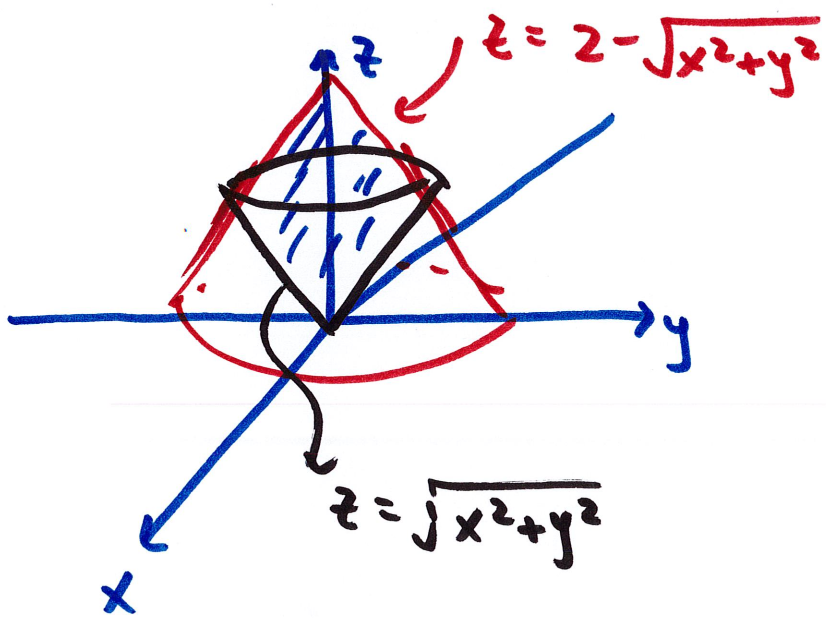
"ceiling" is how far in x we can go
bounded by plane $x + 2y + 3z = 6$

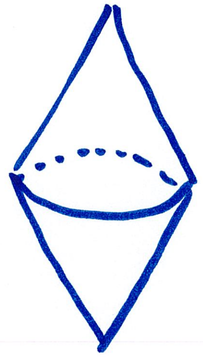
so, $\boxed{0 \le x \le 6 - 2y - 3z}$

order: $dx\,dy\,dz$

$$\underset{z}{\int_0^2} \underset{y}{\int_0^{3-\frac{3}{2}z}} \underset{x}{\int_0^{6-2y-3z}} \overbrace{dx\,dy\,dz}^{dv} = \cdots = \boxed{6}$$

example Volume above $z = \sqrt{x^2+y^2}$ and below $z = 2 - \sqrt{x^2+y^2}$

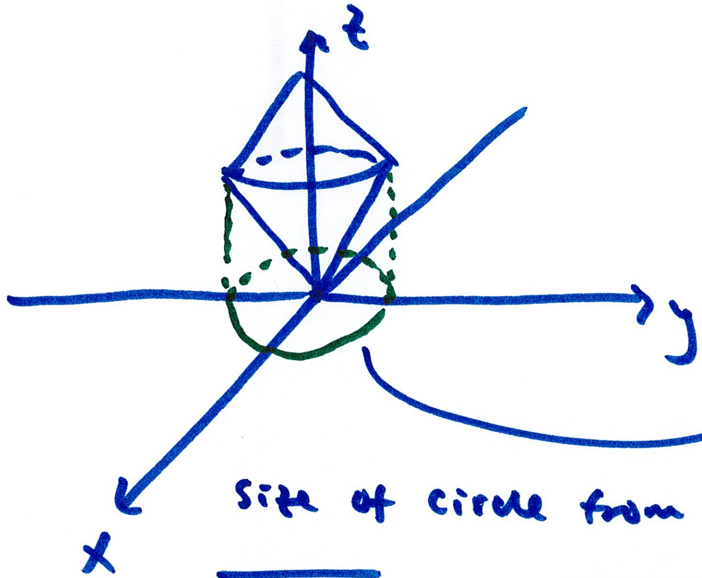


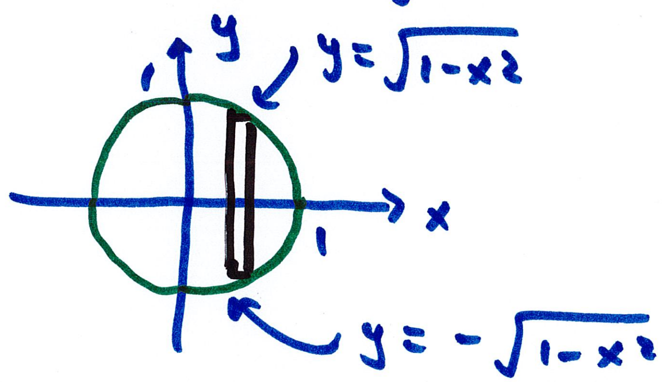


volume = ?

pick a plane to be the "floor"

here, let's pick xy-plane





$\boxed{-1 \le x \le 1}$

$\boxed{-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}}$

size of circle from intersection of cones

$\sqrt{x^2+y^2} = 2 - \sqrt{x^2+y^2}$

$\sqrt{x^2+y^2} = 1 \rightarrow x^2+y^2 = 1$

the "floor" of the solid is the lower cone : lower bound of z

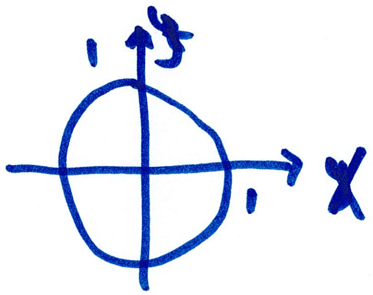
the "ceiling" is the upper cone

$$\boxed{\sqrt{x^2+y^2} \le z \le 2-\sqrt{x^2+y^2}}$$

$$\text{volume} = \int_{\substack{-1 \\ x}}^{1} \int_{\substack{-\sqrt{1-x^2} \\ y}}^{\sqrt{1-x^2}} \int_{\substack{\sqrt{x^2+y^2} \\ z}}^{2-\sqrt{x^2+y^2}} \overbrace{dz\,dy\,dx}^{dV}$$

terrible in Cartesian because "floor" is circular

try expressing bounds for y and x in polar



polar: $0 \le r \le 1$

$0 \le \theta \le 2\pi$

and $dy\,dx = r\,dr\,d\theta$

$r \le z \le 2-r$

$$\int_{\substack{0 \\ \theta}}^{2\pi} \int_{\substack{0 \\ r}}^{1} \int_{\substack{r \\ z}}^{2-r} r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^1 rz \Big|_{z=r}^{z=2-r} dr\,d\theta = \int_0^{2\pi} \int_0^1 (2r-2r^2)\,dr\,d\theta$$

$$= 2\pi \int_0^1 (2r - 2r^2)\,dr$$

$$= 2\pi \left(r^2 - \frac{2}{3} r^3 \right|_0^1 = 2\pi \left(\frac{1}{3} \right) = \boxed{\frac{2\pi}{3}}$$

<u>example</u> Rewrite $\displaystyle\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$

as $\displaystyle\iiint_D dx\,dz\,dy$ and evaluate

as given: $0 \le x \le 1$

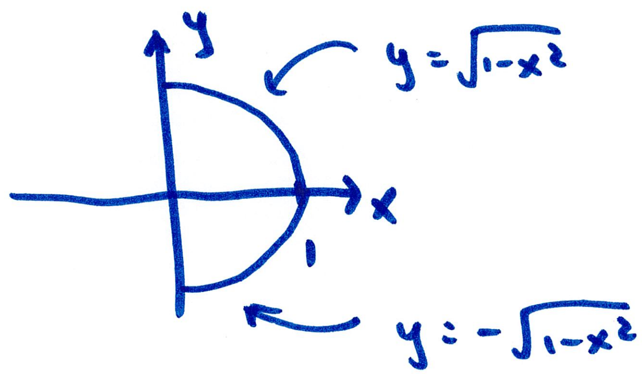
$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$ } "floor" (last two rounds of integration)

$0 \le z \le \sqrt{1-x^2-y^2}$ "ceiling" first round

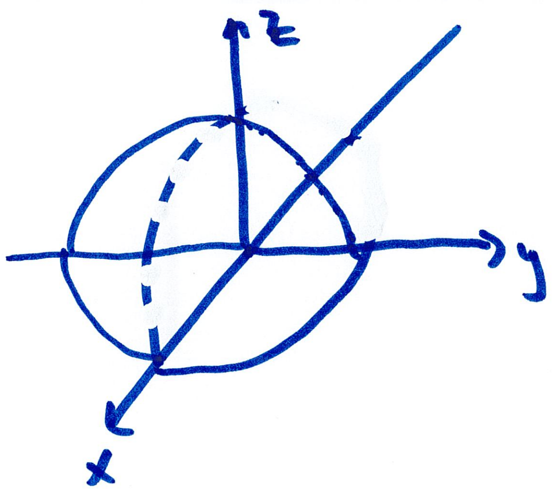
sketch the solid involved: $z = \sqrt{1-x^2-y^2} \longleftrightarrow x^2+y^2+z^2 = 1$

$z = 0$ makes this just the top half

the "floor" shows its shadow



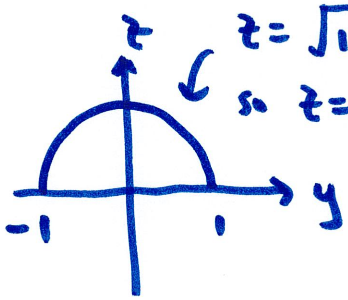
so this means we are only looking at the right half of the upper half of a sphere of radius 1



so, even without doing any integral, we know the volume is $\frac{1}{4} \cdot \frac{4}{3}\pi(1)^3 = \frac{\pi}{3}$

now rewrite order w/ $dx\,\underbrace{dz\,dy}_{\text{"floor"}}$

"floor" w/ y bounded by constants



$z = \sqrt{1-x^2-y^2}$ w/ $x=0$

so $z = \sqrt{1-y^2}$

$\boxed{-1 \le y \le 1}$

$\boxed{0 \le z \le \sqrt{1-y^2}}$

the "ceiling" is how far we can go in x :

$$0 \leq x \leq \sqrt{1-y^2-z^2}$$

$\uparrow$ yz-plane

from [illegible]

$$z = \sqrt{1-x^2-y^2}$$

new integral: $\displaystyle\int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \int_{0}^{\sqrt{1-y^2-z^2}} dx\,dz\,dy = \frac{\pi}{3}$ (from geometry)

$$\underset{y}{} \quad \underset{z}{} \quad \underset{x}{}$$