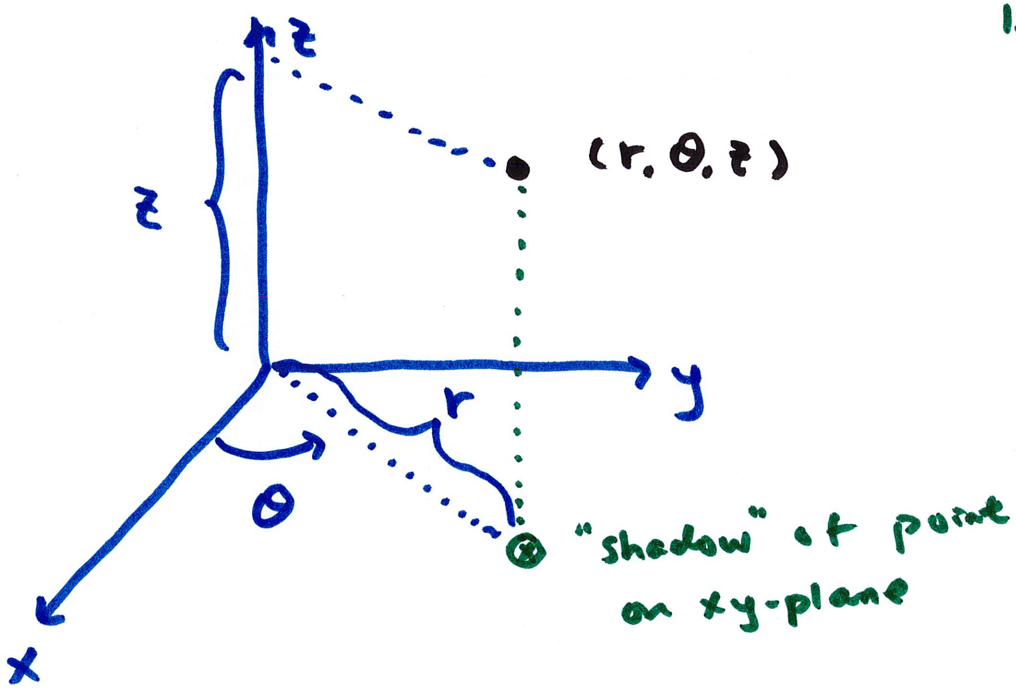
16.5 Triple Integrals in Cylindrical Coordinates

think of cylindrical as a hybrid of polar and Cartesian

in cylindrical, a point is located as $(\underbrace{r, \theta}_{\text{like polar}}, z)$ ↳ Cartesian



shadow of point is located as in polar, then go up z to find point

Conversion: $(x, y, z) \to (r, \theta, z)$

$$\left.\begin{aligned} r^2 &= x^2 + y^2 \\ \tan\theta &= \frac{y}{x} \end{aligned}\right\}$$ just like polar

$$z = z$$

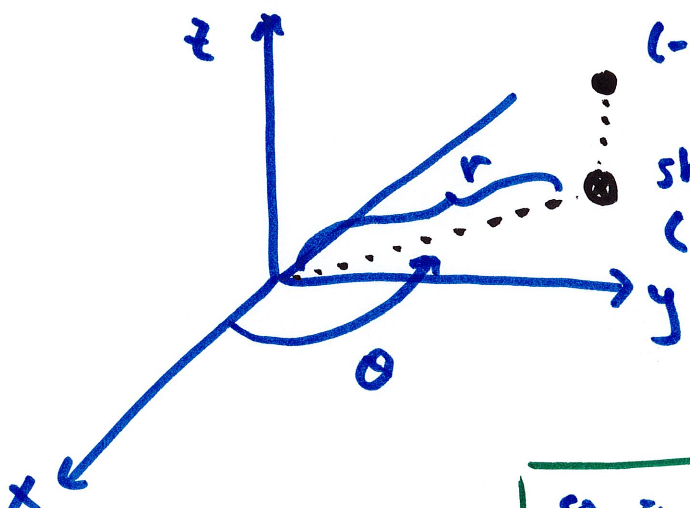
$(r, \theta, z) \to (x, y, z)$

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\}$$ just like polar

$$z = z$$

<u>example</u> $(x, y, z) = (-3, 3\sqrt{3}, 1)$

$(r, \theta, z) = ?$



$$r^2 = x^2 + y^2$$
$$= 9 + 27 = 36$$

$\boxed{r = 6}$ (or $r = -6$)

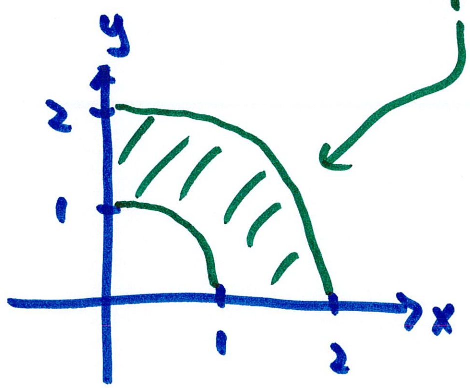
if $r > 0$, θ in QII

$$\tan\theta = \frac{y}{x} = \frac{3\sqrt{3}}{-3} = -\frac{\sqrt{3}}{1}$$
$$= -\frac{\sqrt{3}/2}{1/2} \to \boxed{\theta = \frac{2\pi}{3}}$$

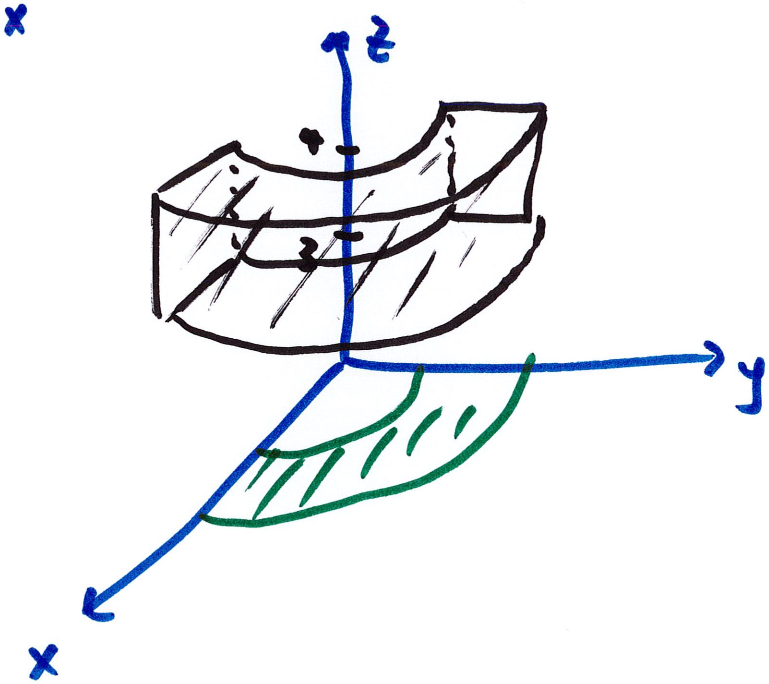
$\boxed{\text{so, in cylindrical: } \left(6, \frac{2\pi}{3}, 1\right)}$

example Sketch the space:

$$\left\{(r,\theta,z):\ \underbrace{1\le r\le 2,\qquad 0\le\theta\le \pi/2,}_{\text{"floor" or shadow of space}}\qquad 3\le z\le 4\right\}$$



then look at z: $3\le z\le 4$



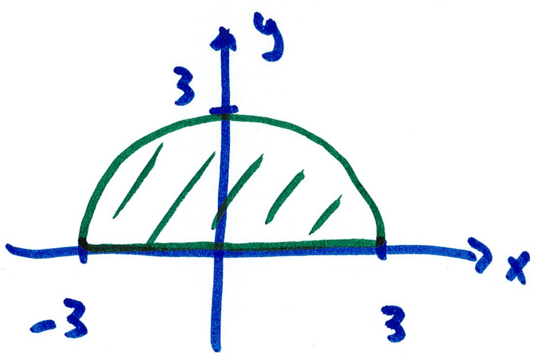
cylindrical is good for cylinders or cylinder-like spaces

<u>example</u> $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\, dz\, dy\, dx$

bad in Cartesian.

Examine the space and see if cylindrical helps.

$$\left.\begin{array}{l} -3 \le x \le 3 \\ 0 \le y \le \sqrt{9-x^2} \end{array}\right\} \text{"floor" of space}$$



the shadow is circle-like so, polar/cylindrical is good

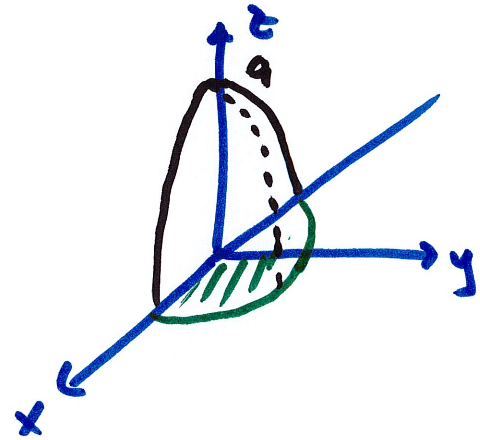
$0 \le r \le 3$

$0 \le \theta \le \pi$

in cylindrical, we keep z as is, but expressed in r, θ

$0 \le z \le 9-x^2-y^2$ paraboloid opening down vertex at $z=9$

$$0 \le z \le 9 - r^2$$



$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} \underbrace{\sqrt{x^2+y^2}}\; dz\; \underbrace{dy\,dx}_{\text{becomes } \underline{r}\,dr\,d\theta}$$

becomes

$$\int_{0}^{\pi}\int_{0}^{3}\int_{0}^{9-r^2} r \cdot dz\,(r\,dr\,d\theta)$$

θ $\quad r \quad$ z

much easier!

$$= \int_{0}^{\pi}\int_{0}^{3}\int_{0}^{9-r^2} r^2\,dz\,dr\,d\theta = \cdots = \boxed{\frac{162\pi}{5}}$$

example

$$\int_0^4 \int_0^{1/\sqrt{2}} \int_x^{\sqrt{1-x^2}} e^{-x^2-y^2}\, dy\, dx\, dz$$

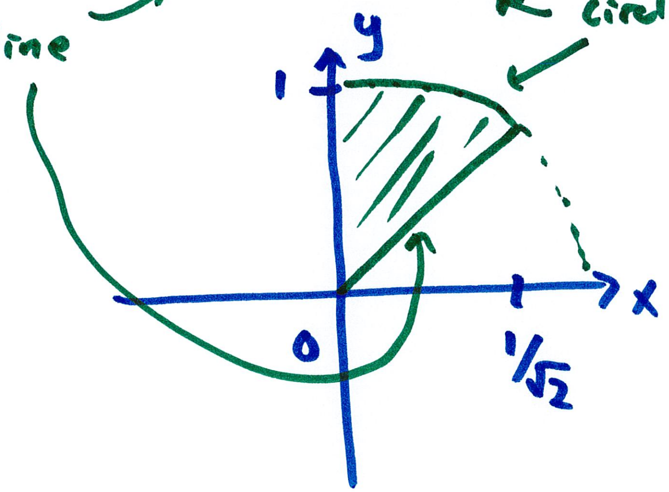
Can't progess with Cartesian

the upper bound of y suggests circle-like shadow

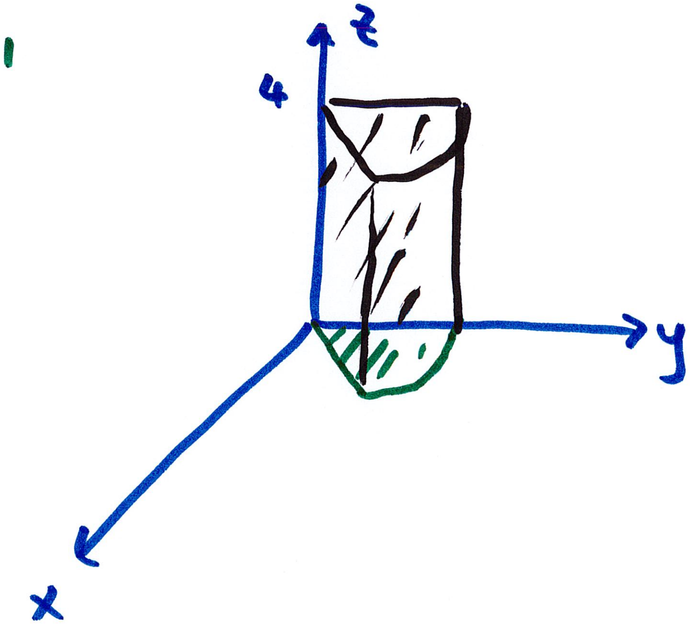
$$\left.\begin{array}{l} 0 \le x \le 1/\sqrt{2} \\ x \le y \le \sqrt{1-x^2} \end{array}\right\} \text{"shadow" of space}$$

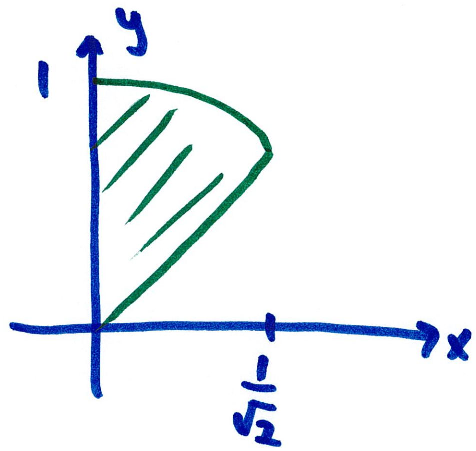
line

circle radius 1



$0 \le z \le 4$





bounds in polar: $0 \le r \le 1$

$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

keep z: $0 \le z \le 4$

$$\int_0^4 \int_0^{1/\sqrt{2}} \int_x^{\sqrt{1-x^2}} e^{-x^2-y^2} \, \underbrace{dy\,dx}_{r\,dr\,d\theta}\,dz \quad \text{becomes}$$

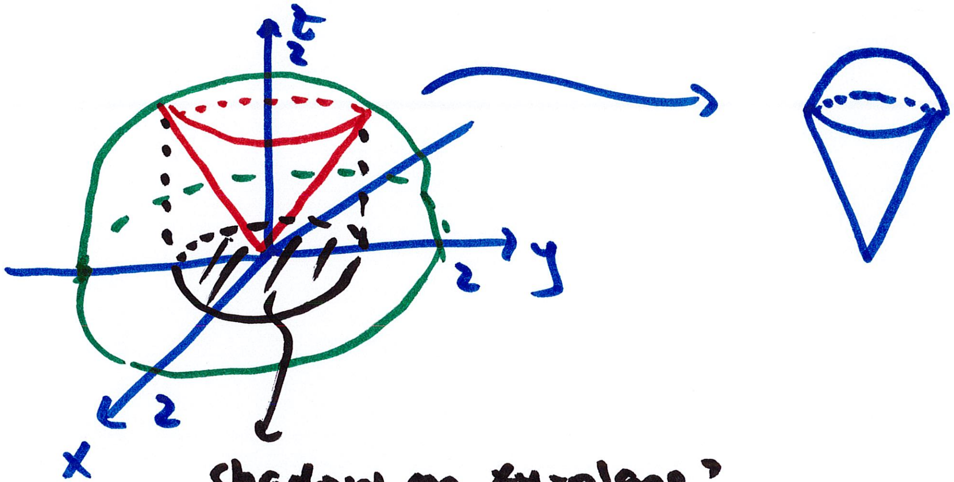
$$= \underset{z}{\int_0^4} \underset{\theta}{\int_{\pi/4}^{\pi/2}} \underset{r}{\int_0^1} e^{-r^2} \cdot r\,dr\,d\theta\,dz$$

$$= \int_0^4 \int_{\pi/4}^{\pi/2} \int_0^1 r e^{-r^2}\,dr\,d\theta\,dz = \cdots = \boxed{\frac{\pi}{2}\left(1-e^{-1}\right)}$$

example Find mass of the solid bounded above $\underbrace{x^2+y^2+z^2=4}_{\text{sphere radius } 2}$

and bounded below by $\underbrace{z=\sqrt{x^2+y^2}}_{\text{cone}}$

with density $\rho(x,y,z)=z$



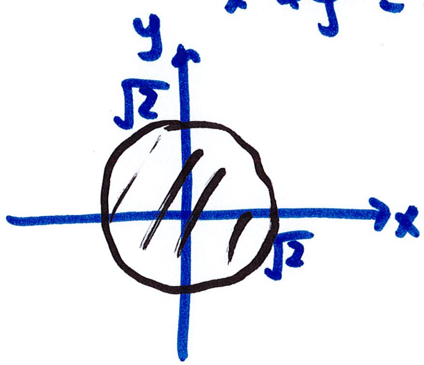
mass = ?

shadow on xy-plane?

sub $z=\sqrt{x^2+y^2}$ into $x^2+y^2+z^2=4$

$x^2+y^2+x^2+y^2=4$

$x^2+y^2=2$ circle radius $\sqrt{2}$



$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$\sqrt{x^2+y^2} \le z \le \sqrt{4-x^2-y^2}$

$\downarrow$

$r \le z \le \sqrt{4-r^2}$

integrate the density $p(x,y,z) = z$ over the space

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} z \quad r \, dz \, dr \, d\theta$$

from $dA = r\,dr\,d\theta$ in polar

$\theta \qquad r \qquad z$

$p(x,y,z)$

$$= \cdots = \boxed{2\pi}$$