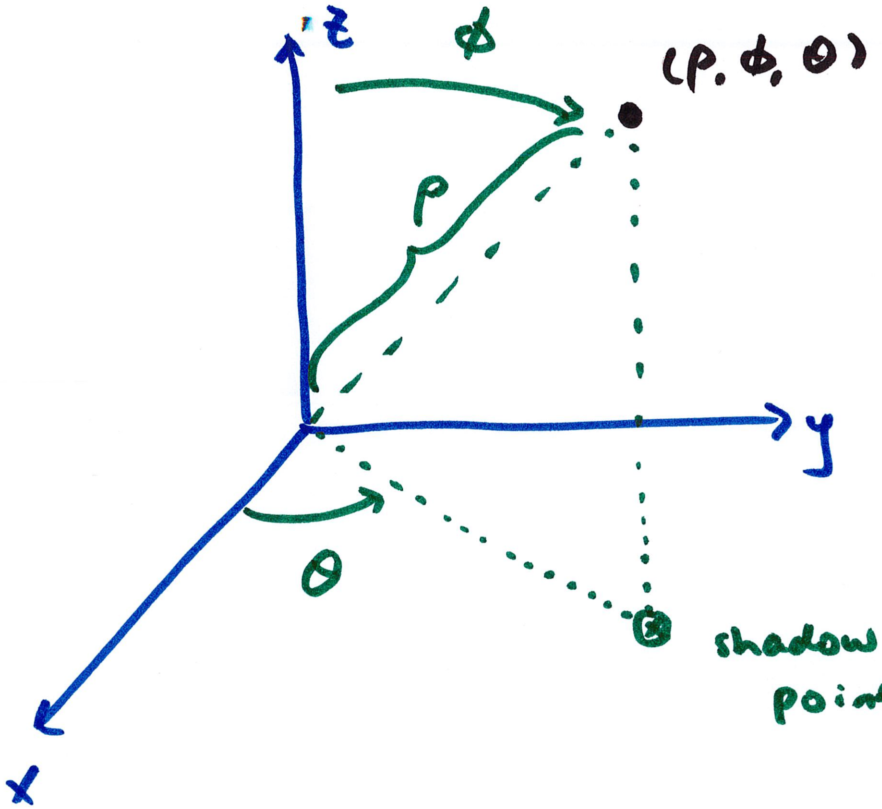
16.5 Triple Integrals in Spherical Coordinates

in spherical, point located by (ρ, ϕ, θ)

same θ as in polar/cylindrical

"rho"
distance from origin

"phi"
measured from positive z-axis down



shadow of point on xy-plane

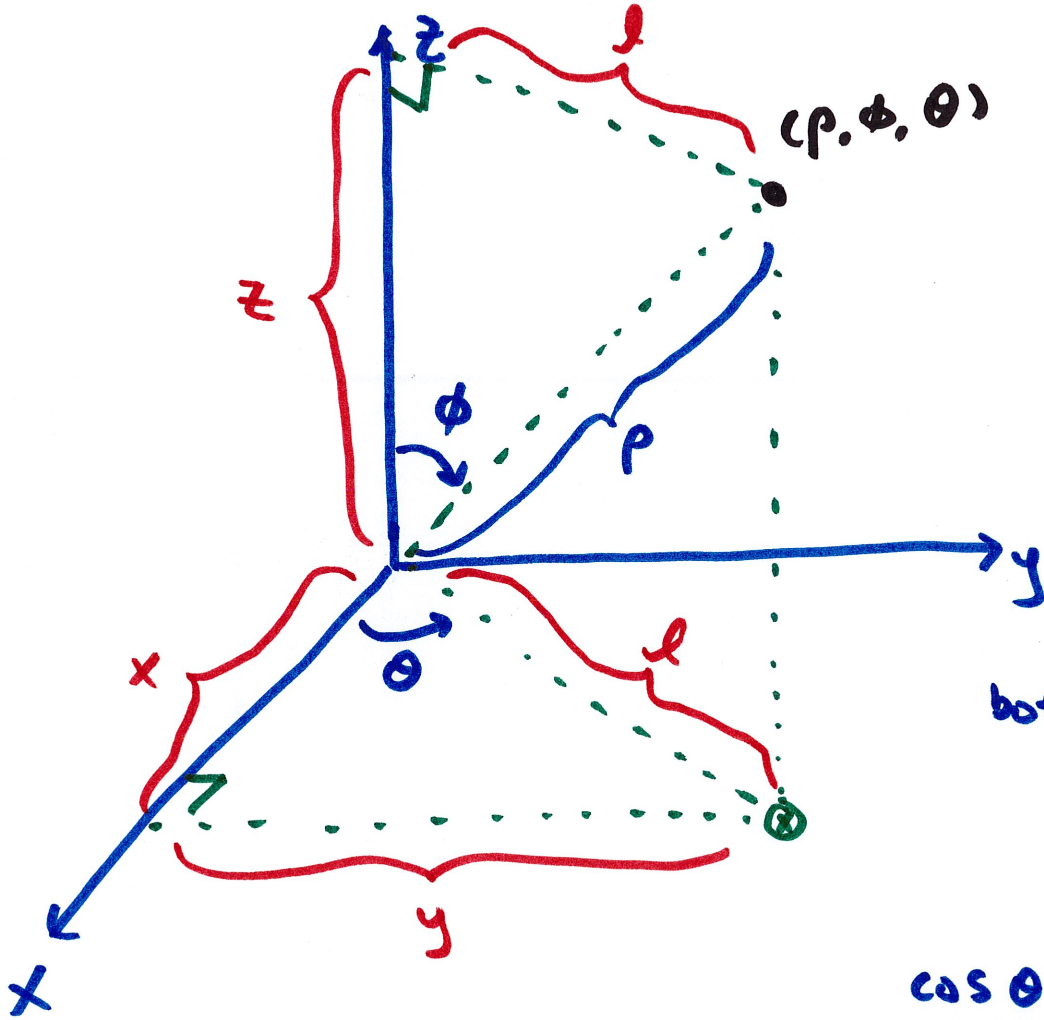
$\rho \geq 0$

$0 \leq \theta \leq 2\pi$

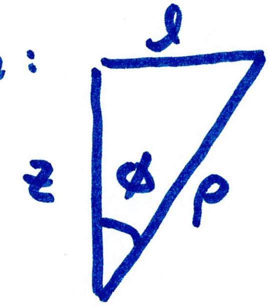
$0 \leq \phi \leq \pi$

we don't need it to go to 2π to cover the entire space because θ allows us to spin all the way around

Converting to/from Cartesian



top triangle:



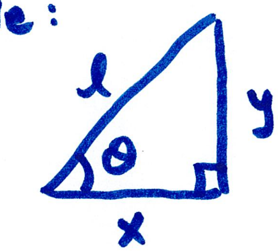
$$\cos\phi = \frac{z}{\rho}$$

so, $\boxed{z = \rho\cos\phi}$

also, $\sin\phi = \frac{l}{\rho}$

$$l = \rho\sin\phi$$

bottom triangle:



$\cos\theta = \frac{x}{l}$ $\quad x = l\cos\theta$

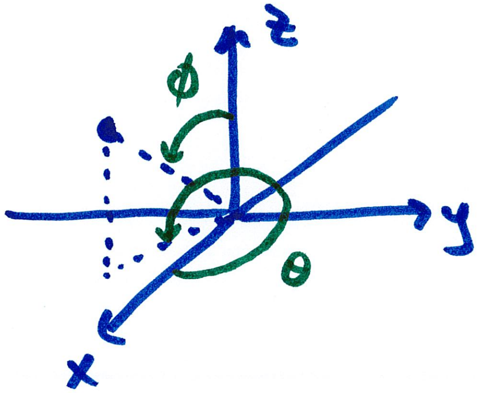
$$\boxed{x = \rho\sin\phi\cos\theta}$$

$\sin\theta = \frac{y}{l}$ $\quad y = l\sin\theta$

$$\boxed{y = \rho\sin\phi\sin\theta}$$

$\boxed{\text{note: } x^2 + y^2 + z^2 = \rho^2}$

<u>example</u> $(x, y, z) = (1, -1, \sqrt{2})$ $(\rho, \phi, \theta) = ?$



got the quadrants right!

by inspection, $\frac{3\pi}{2} \le \theta \le 2\pi$

$0 \le \phi \le \pi/2$

ρ is easy: $\rho^2 = x^2 + y^2 + z^2 = 1 + 1 + 2 = 4$ $\boxed{\rho = 2}$

from $z = \rho \cos\phi$

$\sqrt{2} = 2\cos\phi$ $\cos\phi = \frac{\sqrt{2}}{2}$ $\boxed{\phi = \frac{\pi}{4}}$

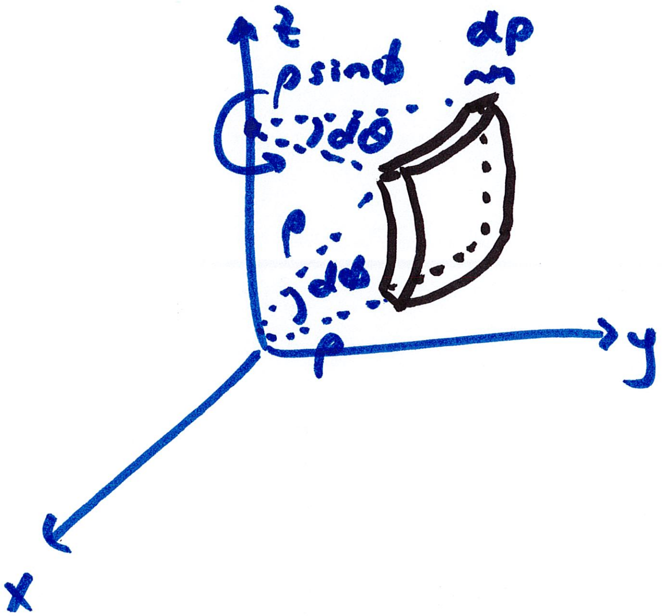
$$\left.\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \end{aligned}\right\} \quad \frac{y}{x} = \tan\theta = \frac{-1}{1} = -1$$

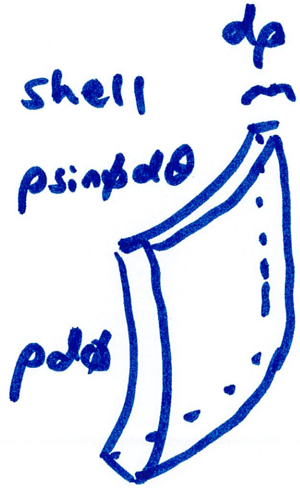
$$\theta = \tan^{-1}(-1) = \frac{7\pi}{4}$$

$$\boxed{\theta = \frac{7\pi}{4}}$$

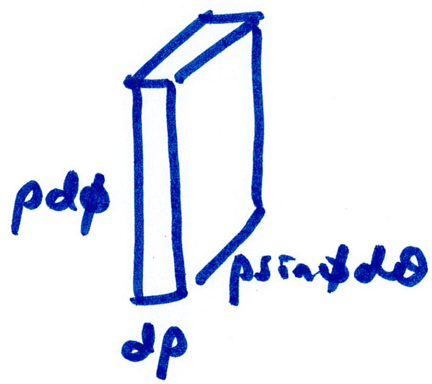
dV in spherical is complicated

imagine a small piece of a spherical shell





when dρ, dφ, dθ are small, the shell is rough a rectangular box



so, $\boxed{dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$

example

$$\int_0^6 \int_0^{\sqrt{36-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{72-x^2-y^2}} dz\, dy\, dx$$

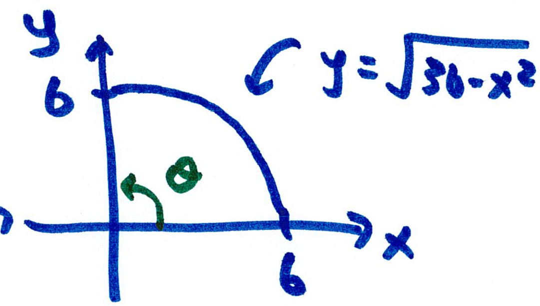
terrible in Cartesian

z's upper bound is part of a sphere → clue to go to spherical

"floor":

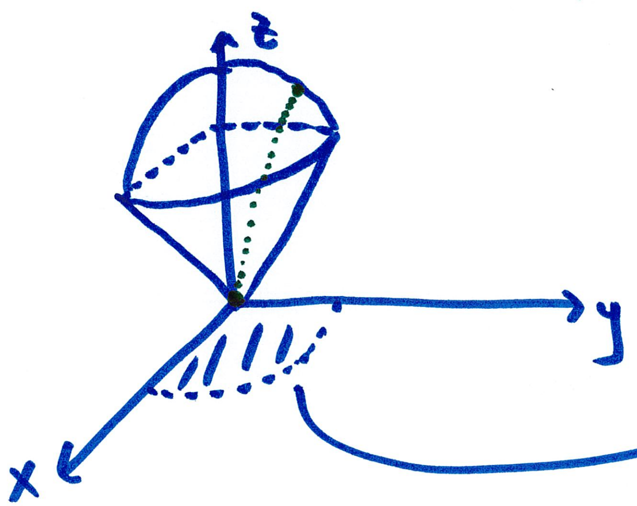
$0 \le x \le 6$

$0 \le y \le \sqrt{36-x^2}$



$$\underbrace{\sqrt{x^2+y^2}}_{\text{cone}} \le z \le \underbrace{\sqrt{72-x^2-y^2}}_{\text{~~radiu~~ sphere radius } \sqrt{72}}$$

so we see $\boxed{0 \le \theta \le \frac{\pi}{2}}$



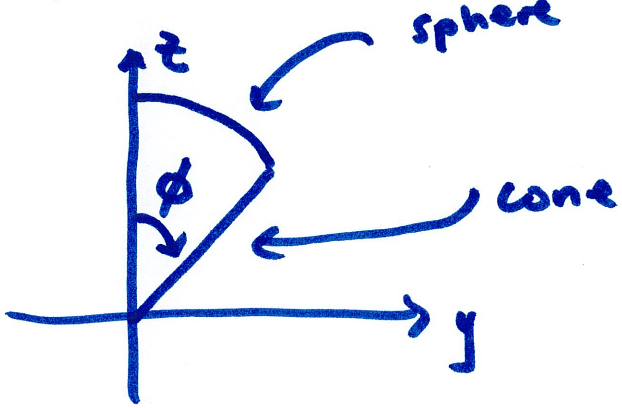
draw a line from origin to edge of volume to see ρ bounds

ρ is from origin to sphere of radius $\sqrt{72}$

so, $\boxed{0 \le \rho \le \sqrt{72}}$

now ϕ bounds

look at yz-trace



cone equation: $z = \sqrt{x^2+y^2}$

on yz-plane, $z = \sqrt{y^2} = y$

slope is 1, so angle is $\frac{\pi}{4}$

$\boxed{0 \le \phi \le \pi/4}$

now the new integral:

$$\underset{\theta}{\int_0^{\pi/2}} \underset{\phi}{\int_0^{\pi/4}} \underset{\rho}{\int_0^{\sqrt{72}}} \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV} = \cdots = \boxed{12\sqrt{72}\,\pi\left(1 - \frac{1}{\sqrt{2}}\right)}$$

example Find volume of ~~of~~ solid outside $\rho = 1$ and inside $\rho = 2\cos\phi$

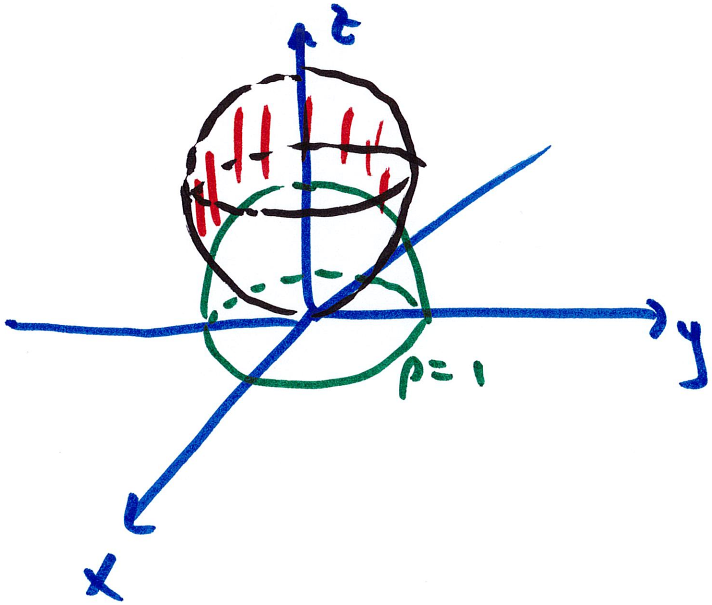
sphere radius 1 centered at $(0,0,0)$

sphere radius 1 centered at $(0,0,1)$

$$\rho = 2\cos\phi$$

$$\sqrt{x^2+y^2+z^2} = \frac{2\rho\cos\phi}{\rho} = \frac{2z}{\sqrt{x^2+y^2+z^2}}$$

$$x^2+y^2+z^2 = 2z \quad \cdots \quad \to \quad x^2+y^2+(z-1)^2 = 1$$

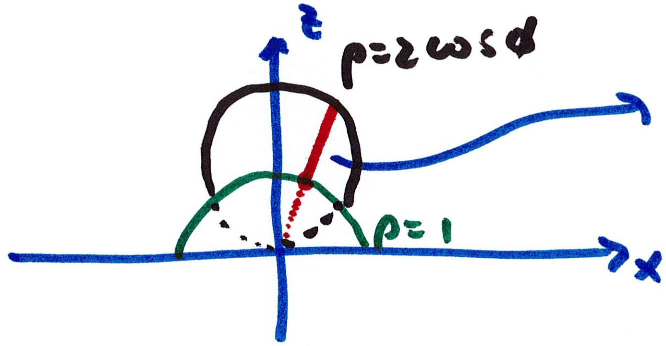


volume of space in between (like Toad's hat)

by inspection, $\boxed{0 \le \theta \le 2\pi}$

volume of his hat?

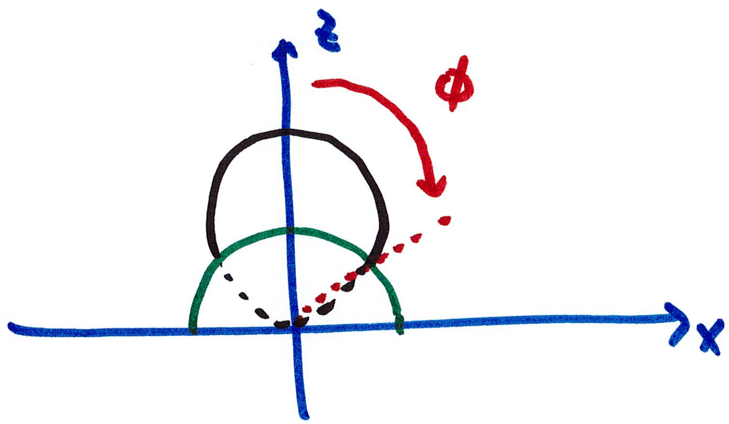
ρ bounds:



ρ starts at lower circle/sphere
ends at upper one

$$\boxed{1 \le \rho \le 2\cos\phi}$$

ϕ bounds:



intersection of spheres

$\rho = 1$

$\rho = 2\cos\phi$

$1 = 2\cos\phi \rightarrow \phi = \frac{\pi}{3}$

so $\boxed{0 \le \phi \le \frac{\pi}{3}}$

volume: $$\underbrace{\int_0^{2\pi}}_{\theta} \underbrace{\int_0^{\pi/3}}_{\phi} \underbrace{\int_1^{2\cos\phi}}_{\rho} \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV} = \dots = \boxed{\frac{11\pi}{12}}$$