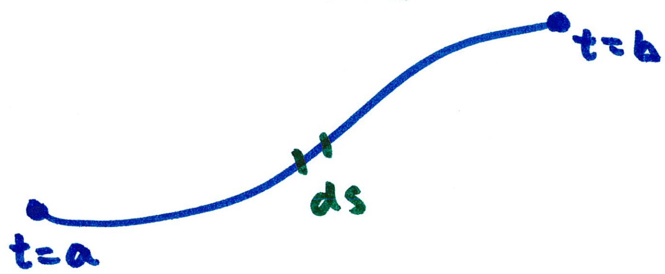
17.2 (part 1) Line Integrals of Functions

remember length of $\vec{r}(t)$, $a \le t \le b$ is

$$L(t) = \int_a^b \underbrace{|\vec{r}\,'(t)|\,dt}_{ds} = \int_C ds$$

ds
small chunk of
an arc

$\hookrightarrow$ C : parametrization of $\vec{r}(t)$



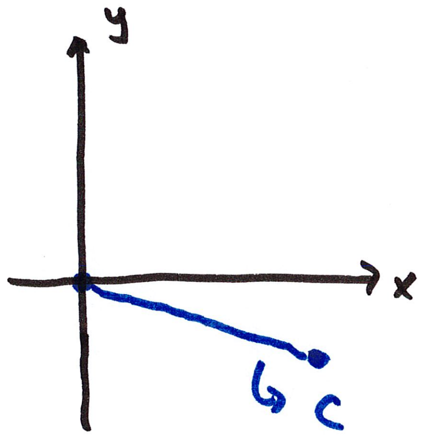
generalize a bit : integrate some $f(x,y)$ along with ds

line integral : $\displaystyle\int_C f(x,y)\,ds$

if $f(x,y)$ is density of wire in shape of C, then $\int_C f(x,y)\,ds$ gives the mass of wire in 3D. $\int_C f(x,y,z)\,ds$

parametrization of the curve is an important process

example $\int_C xe^{y^2}\,ds$ C: line segment from $(0,0)$ to $(4,-1)$



the curve here is a line

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

$$= \langle 0, 0\rangle + t\langle 4, -1\rangle \qquad 0 \le t \le 1$$

$$\vec{r}(t) = \langle \underset{x}{4t}, \underset{y}{-t} \rangle$$

$$ds = |\vec{r}\,'(t)|\,dt = |\langle 4, -1\rangle|\,dt = \sqrt{17}\,dt$$

$$\int_C xe^{\overset{-t}{y^2}}\,ds = \int_0^1 4t\,e^{(-t)^2}\sqrt{17}\,dt = 4\sqrt{17}\int_0^1 te^{t^2}\,dt$$

x of parametrization of C here, 4t

$$= \dots = \boxed{2\sqrt{17}\,(e-1)}$$

note $\vec{r}(t) = \langle 4t, -t \rangle$ ~~is no~~ $0 \le t \le 1$ is not the only way

~~$0 \le t \le 1$~~

to describe this curve

we could use $\vec{r}(t) = \langle 8t, -2t \rangle$ $\quad 0 \le t \le \frac{1}{2}$

would that change the integral $\int_C f(x,y)\, ds$?

$$\int_C x e^{y^2}\, ds \qquad C: \vec{r}(t) = \langle 8t, -2t \rangle \quad 0 \le t \le \frac{1}{2}$$

$$ds = |\vec{r}\,'(t)|\, dt = |\langle 8, -2 \rangle|\, dt = \sqrt{68}\, dt$$

$$= \int_0^{1/2} (8t) e^{(-2t)^2} \sqrt{68}\, dt = 8\sqrt{68} \int_0^{1/2} t\, e^{4t^2}\, dt \qquad \begin{aligned} u &= 4t^2 \\ du &= 8t\, dt \end{aligned}$$

$$= 8\sqrt{68} \int_0^1 \frac{1}{8} e^u\, du = \sqrt{68} \int_0^1 e^u\, du = \sqrt{68}\,(e^1 - 1) = \boxed{2\sqrt{17}(e-1)}$$

$4 \cdot 17$ $\qquad$ same

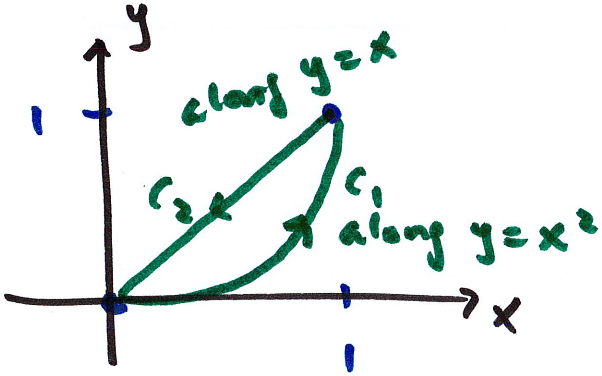
so, the parametrization choice does NOT affect the result

example

$$\int_C (x+\sqrt{y})\,ds$$

C: from $(0,0)$ to $(1,1)$ along $y=x^2$

then $(1,1)$ to $(0,0)$ along $y=x$



need two parametrizations

C_1: * $\vec{r}(t) = \langle t, t^2 \rangle$ $\quad 0 \le t \le 1$

$\quad\quad x \quad y$

* along $y=x^2$

so $y=x^2 \rightarrow t^2 = (t)^2$

or $\vec{r}(t) = \langle 2t, 4t^2 \rangle$ $\quad 0 \le t \le 1/2$

C_2: * $\vec{r}(t) = \langle 1-t, 1-t \rangle$ $\quad 0 \le t \le 1$

$\vec{r}(t) = \langle t, t \rangle$ $\quad t$ from 1 to 0

C_1 : $\vec{r}(t) = \langle t, t^2 \rangle$ $\quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle 1, 2t \rangle$ so $ds = |\vec{r}\,'(t)|\,dt = \sqrt{1+4t^2}\,dt$

C_2 : $\vec{r}(t) = \langle 1-t, 1-t \rangle$ $\quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle -1, -1 \rangle$ so $ds = |\vec{r}\,'(t)|\,dt = \sqrt{2}\,dt$

$$\int_C (x+\sqrt{y})\,ds$$

$$= \underbrace{\int_0^1 \underbrace{(t+\sqrt{t^2})}_{\substack{\text{using } x,y \\ \text{of } \vec{r}(t) \text{ on } C_1}} \sqrt{1+4t^2}\,dt}_{C_1} + \underbrace{\int_0^1 \Big[\underbrace{(1-t)+\sqrt{1-t}}_{\substack{\text{using } x,y \\ \text{of } \vec{r} \text{ on } C_2}}\Big] \sqrt{2}\,dt}_{C_2}$$

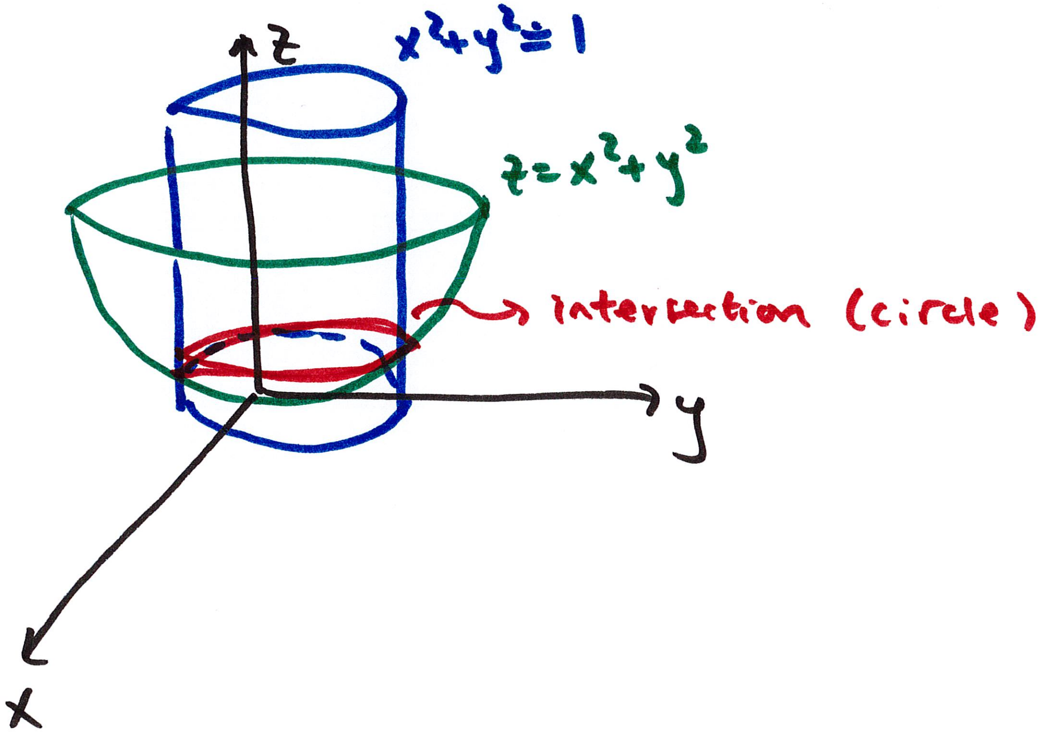
$$= \cdots = \boxed{\frac{1}{6}(5\sqrt{5}-1) + \frac{7}{3\sqrt{2}}}$$

__example__

$$\int_C (x+y+z)\,ds$$

C : intersection of paraboloid $z=x^2+y^2$ and the cylinder $x^2+y^2=1$

going counterclockwise when viewed from above



circle size :

$$z=x^2+y^2$$
$$x^2+y^2=1$$
$$z=1$$

at $z=1$, $z=x^2+y^2$

$x^2+y^2=1$

circle radius 1

C: is this circle

now parametrize C : $\vec{r}(t) = \langle \underbrace{\cos t, \sin t}_{\text{circle radius 1}}, 1 \rangle \quad 0 \le t \le 2\pi$

intersection of paraboloid and cylinder is at $z=1$

or $\vec{r}(t) = \langle \cos(2t), \sin(2t), 1 \rangle \quad 0 \le t \le \pi$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 0 \rangle$

$ds = |\vec{r}\,'(t)|\,dt = 1 \cdot dt = dt$

$$\int_C (x+y+z)\,ds$$

$$= \int_0^{2\pi} (\cos t + \sin t + 1)\,dt = \cdots = \boxed{2\pi}$$