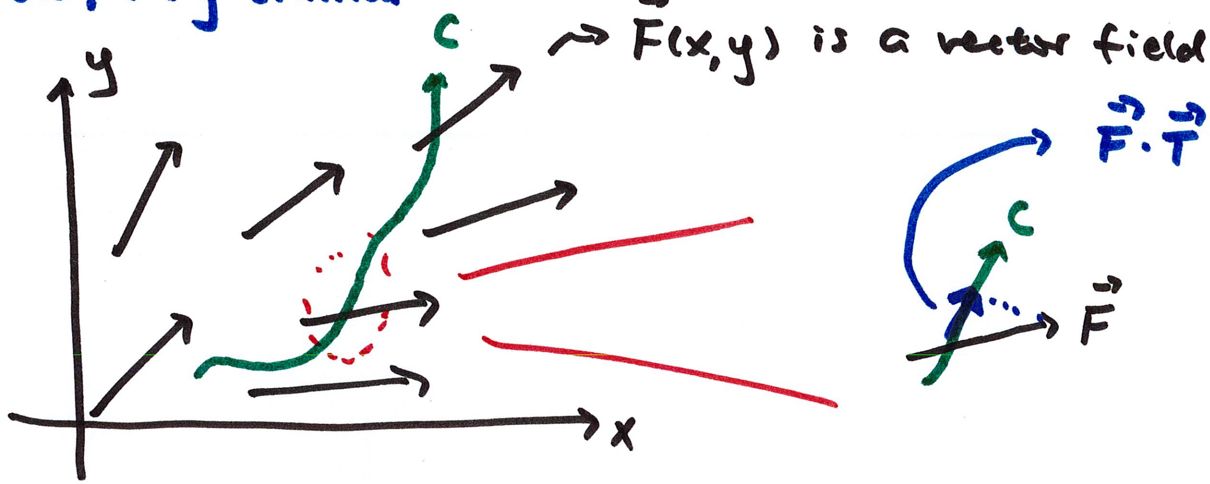
17.2 (part 2) Line Integrals in Vector Fields

last time: line integral in scalar field $\int_C f\,ds$

in vector field, very similar



we want to accumulate certain components of $\vec{F}$ along the path

typically along the motion of the path C

$$\int_C \vec{F}\cdot\vec{T}\,ds$$

ds: small part of the path

$\vec{T}$: unit tangent

$$\boxed{\int_C \vec{F}\cdot\vec{T}\,ds}$$

C: parametrized as $\vec{r}(t)$, $a \le t \le b$

then $\vec{T} = \dfrac{\vec{r}\,'}{|\vec{r}\,'|}$ and we remember $ds = |\vec{r}\,'|\,dt$

$\int_C \vec{F}\cdot\vec{T}\,ds$ becomes

$$\int_C \vec{F}\cdot\frac{\vec{r}\,'}{|\vec{r}\,'|}|\vec{r}\,'|\,dt = \boxed{\int_C \vec{F}\cdot\vec{r}\,'\,dt}$$

another expression of the same integral

$$\vec{r}\,'dt = \frac{d\vec{r}}{dt}dt = d\vec{r}$$

so, another equivalent form: $\boxed{\int_C \vec{F}\cdot d\vec{r}}$

common application: work done by force vector field $\vec{F}$ in moving something along C basic process begins with a parametrization of C

example $\vec{F} = \langle xy, y-x \rangle$

C: line segment from $(0,1)$ to $(2,4)$

parametrize C: $\vec{r}(t) = \langle 0,1 \rangle + t\langle 2,3 \rangle \quad 0 \le t \le 1$

$\vec{r}(t) = \langle 2t, 1+3t \rangle \quad 0 \le t \le 1$

again, we can choose to parametrize in any way we want

$\vec{r}(t) = \langle 4t, 1+6t \rangle \quad 0 \le t \le 1/2$

is also ok.

let's use $\vec{r}(t) = \langle 2t, 1+3t \rangle \quad 0 \le t \le 1$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{r}\,'\,dt = \int_C \vec{F}\cdot d\vec{r}$$

middle one looks easiest to use

$$\vec{r}\,' = \langle 2, 3 \rangle$$

x y from $\vec{r}(t)$

$$\vec{F} = \langle xy, y - x \rangle = \langle (2t)(3t+1), (3t+1) - (2t) \rangle$$

$$= \langle 6t^2 + 2t, t + 1 \rangle$$

$$\int_C \vec{F} \cdot \vec{r}\,' dt = \int_0^1 \langle 6t^2 + 2t, t+1 \rangle \cdot \langle 2, 3 \rangle \, dt$$

$$= \int_0^1 (12t^2 + 4t + 3t + 3) \, dt = \dots = \boxed{\frac{25}{2}}$$

if C is a closed loop (same starting and end points),

then the line integral $\int_C \vec{F}\cdot\vec{T}\,ds$ is also called the <u>circulation</u>

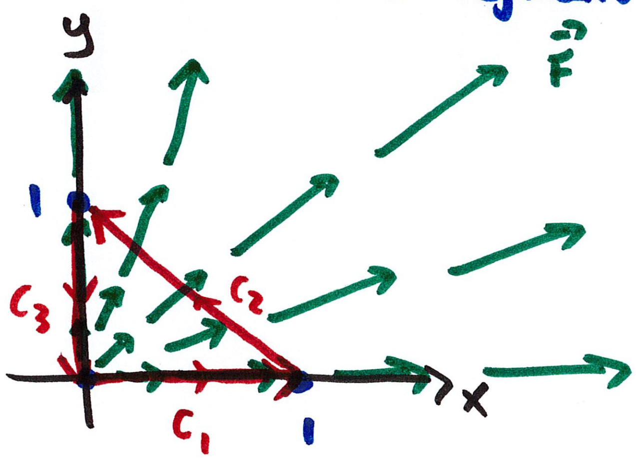
of $\vec{F}$ on C.

<u>example</u> $\vec{F} = \langle x, y\rangle$

C: line segment from $(0,0)$ to $(1,0)$

then line segment from $(1,0)$ to $(0,1)$

then line segment from $(0,1)$ to $(0,0)$



C_1 : $\vec{F}(t) = \langle t, 0\rangle \quad 0 \le t \le 1$

C_2 : $\cancel{\vec{F}(t) = \langle t, 1-t\rangle \quad 0\le t\le 1}$

$\vec{F}(t) = \langle 1-t, t\rangle \quad 0 \le t \le 1$

C_3 : $\vec{F}(t) = \langle 0, 1-t\rangle \quad 0 \le t \le 1$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{r}\,'\,dt = \int_C \vec{F}\cdot d\vec{r}$$

$$= \underbrace{\int_0^1 \underbrace{\langle t, 0\rangle}_{\substack{\vec{F} \text{ w/} \\ x,y \text{ of} \\ \vec{r}(t) \text{ on } C_1}} \cdot \underbrace{\langle 1, 0\rangle}_{\vec{r}\,' \text{ on } C_1}\,dt}_{C_1} + \underbrace{\int_0^1 \langle 1-t, t\rangle \cdot \langle -1, 1\rangle\,dt}_{C_2} + \underbrace{\int_0^1 \langle 0, 1-t\rangle \cdot \langle 0, -1\rangle\,dt}_{C_3}$$

$$= \int_0^1 t\,dt + \int_0^1 (2t-1)\,dt + \int_0^1 (t-1)\,dt = \cdots = \boxed{0}$$

if $\vec{F} = \langle f, g \rangle$

and $\vec{r}(t) = \langle x(t), y(t) \rangle \quad a \le t \le b$

then $\vec{T} = \dfrac{\vec{F}'}{|\vec{F}'|} = \dfrac{\langle x', y' \rangle}{\sqrt{(x')^2 + (y')^2}}$ $\qquad ds = |\vec{r}'|\,dt = \sqrt{(x')^2 + (y')^2}\,dt$

sub into $\displaystyle\int_C \vec{F} \cdot \vec{T}\, ds$

$$= \int_C \langle f, g \rangle \cdot \frac{\langle x', y' \rangle}{\cancel{\sqrt{(x')^2 + (y')^2}}} \cancel{\sqrt{(x')^2 + (y')^2}}\, dt$$

$$= \int_C (f x' + g y')\, dt = \int_C f \underset{\frac{dx}{dt}}{\underset{\downarrow}{x'}} dt + g \underset{\frac{dy}{dt}}{\underset{\downarrow}{y'}} dt = \boxed{\int_C f\, dx + g\, dy}$$

one more equivalent form

so $\int_C f\,dx + g\,dy$ tells us that $\vec{F} = \langle f, g \rangle$ path $\vec{r}(t) = \langle x, y \rangle$

for example, $\int_C xy\,dx + (x+y)\,dy$

C: $(0,0)$ to $(1,1)$ along $y = x^2$

interpret this as $\vec{F} = \langle xy, x+y \rangle$

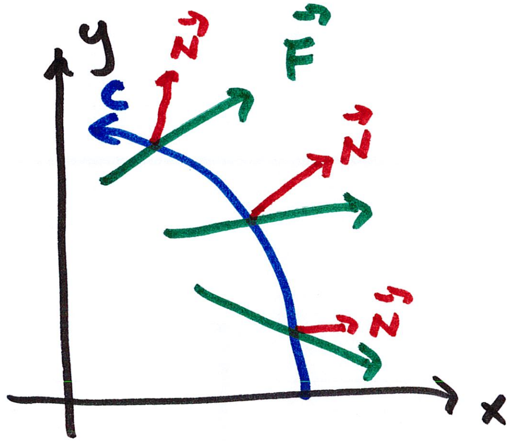
path: $\vec{r}(t) = \langle \underset{x(t)}{t}, \underset{y(t)}{t^2} \rangle \quad 0 \le t \le 1$

then calculate as in previous examples

ANS: $\dfrac{17}{12}$

if we use the unit normal vector $\vec{N}$ instead of unit tangent $\vec{T}$,

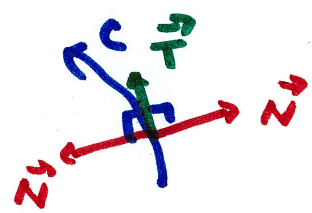
then the line integral $\int_C \vec{F}\cdot\vec{N}\,ds$ is called the <u>**flux integral**</u>



$\int_C \vec{F}\cdot\vec{N}\,ds$ accumulates the portion of $\vec{F}$ flowing <u>through</u> the path

(think of it as water/air flowing through some barrier in the shape of C)

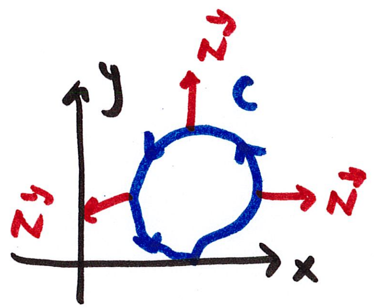
there are two unit normals:



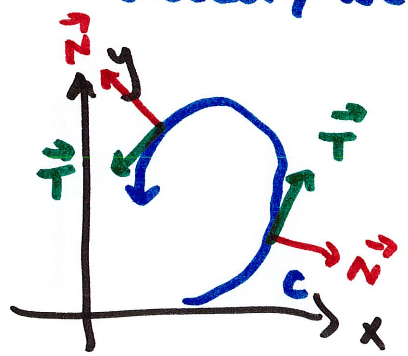
both perpendicular to $\vec{T}$

which one to use?

by convention, if C is a closed loop, we choose $\vec{N}$ to point out



if C is not closed, we choose $\vec{N}$ to point to the right of $\vec{T}$



$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|}$ or $\vec{N} = \frac{-\vec{T}'}{|\vec{T}'|}$ whichever is consistent with the above