17.3 Conservative Vector Fields and the Fundamental Theorem of Line Integrals

if $\vec{F}$ is conservative, then $\vec{F} = \vec{\nabla}\phi$ $\quad$ ϕ: potential function

if given ϕ, finding $\vec{F}$ is easy.

harder: given $\vec{F}$, how to check if conservative, and if so, how to find ϕ.

let $\vec{F} = \langle f, g \rangle$ be a conservative vector field

then we know $\vec{F} = \langle f, g \rangle = \vec{\nabla}\phi = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right\rangle$

so $f = \frac{\partial \phi}{\partial x}$

$g = \frac{\partial \phi}{\partial y}$

we know the equality of partial derivatives: $\frac{\partial}{\partial y}\Big(\underbrace{\frac{\partial \phi}{\partial x}}_{f}\Big) = \frac{\partial}{\partial x}\Big(\underbrace{\frac{\partial \phi}{\partial y}}_{g}\Big)$

So, this means, if $\vec{F} = \langle f, g \rangle$ is conservative, then $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ or $f_y = g_x$

<u>example</u> $\vec{F} = \langle x, y \rangle$ conservative ?

$\qquad f \qquad\qquad g$

is $f_y = g_x$?

$$\frac{\partial}{\partial y}(x) = 0 \qquad \frac{\partial}{\partial x}(y) = 0$$

so, yes, $f_y = g_x \rightarrow \vec{F}$ is conservative.

<u>example</u> $\vec{F} = \langle -y, x \rangle$

$\qquad f \qquad\qquad g$

is $f_y = g_x$?

$f_y = -1, \quad g_x = 1$

$f_y \neq g_x$ so $\vec{F}$ is <u>not</u> conservative how to find ϕ if $\vec{F}$ is conservative ($\vec{F} = \vec{\nabla}\phi$)?

<u>example</u> $\vec{F} = \langle \underbrace{x+y}_{f}, \underbrace{x}_{g} \rangle$

$f_y = 1$, $g_x = 1$ so $\vec{F}$ is conservative.

now find ϕ

$$\vec{F} = \langle x+y, x \rangle = \vec{\nabla}\phi = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right\rangle$$

so, $x+y = \frac{\partial \phi}{\partial x}$ ①

$x = \frac{\partial \phi}{\partial y}$ ②

from ①, integrate with respect to x

$$\phi = \int (x+y)\, dx = \frac{1}{2}x^2 + xy + \underbrace{a(y)}$$ ③

constant

integrate with respect to x

y → constant

function that ~~dep~~ can contain y because it is a "constant" when we took partial with respect to x to get $\frac{\partial \phi}{\partial x}$

③: $\phi = \frac{1}{2}x^2 + xy + a(y)$

take the partial with respect to y and compare to ②

$$\frac{\partial \phi}{\partial y} = x + \frac{da}{dy} = \underbrace{x}_{②}$$

so, $\frac{da}{dy} = 0 \rightarrow a = c$ a true constant

so, $\boxed{\phi = \frac{1}{2}x^2 + xy + c}$

check: does $\vec{\nabla}\phi = \vec{F} = \langle x+y, x \rangle$?

$\vec{\nabla}\phi = \langle x+y, x \rangle \nearrow$ yes.

3D vector field : $\vec{F} = \langle f, g, h \rangle$ how to check if conservative and find ϕ?

if conservative, then $\vec{F} = \langle f, g, h \rangle = \vec{\nabla}\phi = \langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \rangle$

so
$$f = \frac{\partial \phi}{\partial x} = \phi_x$$
$$g = \frac{\partial \phi}{\partial y} = \phi_y$$
$$h = \frac{\partial \phi}{\partial z} = \phi_z$$

we also know
$$(\overset{f}{\phi_x})_y = (\overset{g}{\phi_y})_x$$
$$(\overset{g}{\phi_y})_z = (\overset{h}{\phi_z})_y$$
$$(\overset{h}{\phi_z})_x = (\overset{f}{\phi_x})_z$$

$$\boxed{\begin{aligned} f_y &= g_x \\ g_z &= h_y \\ h_x &= f_z \end{aligned}}$$

if all true, $\vec{F} = \langle f, g, h \rangle$ is conservative example Is $\vec{F} = \langle \underset{f}{x^2 - ze^y}, \underset{g}{y^3 - xze^y}, \underset{h}{z^4 - xe^y} \rangle$ conservative?

if so, find ϕ $(\vec{F} = \vec{\nabla}\phi)$

is $f_y = g_x$? $\quad -ze^y = -ze^y$ yes

$f_z = h_x$? $\quad -e^y = -e^y$ yes

$g_z = h_y$? $\quad -xe^y = -xe^y$ yes

so, $\vec{F}$ is conservative

now find ϕ $\quad \vec{F} = \langle \phi_x, \phi_y, \phi_z \rangle$

$\phi_x = x^2 - ze^y$ ①

$\phi_y = y^3 - xze^y$ ②

$\phi_z = z^4 - xe^y$ ③

from 1 : $\phi = \int (x^2 - ze^y)\,dx = \frac{1}{3}x^3 - xze^y + \underbrace{a(y,z)}_{\text{function that can contain } y,z}$ 4

dx = others are constants

take partial with respect to y, compare to 2

$$\phi_y = -xze^y + \frac{\partial a}{\partial y} = \underbrace{y^3 - xze^y}_{2} \qquad \text{so } \frac{\partial a}{\partial y} = y^3 \quad 5$$

take partial of 4 with z, compare to 3

$$\phi_z = -xe^y + \frac{\partial a}{\partial z} = \underbrace{z^4 - xe^y}_{4} \qquad \text{so } \frac{\partial a}{\partial z} = z^4 \quad 6$$

integrate 5 with y $\rightarrow a = \frac{1}{4}y^4 + b(z)$

take partial with z, compare to 6

$$\frac{\partial a}{\partial z} = \frac{db}{dz} = z^4 \rightarrow b = \frac{1}{5}z^5 + C \qquad \text{so, } a = \frac{1}{4}y^4 + \frac{1}{5}z^5 + C$$

so, $\boxed{\phi = \frac{1}{3}x^3 - xze^y + \frac{1}{4}y^4 + \frac{1}{5}z^5 + C}$

why bother with ϕ?

because if $\vec{F}$ is conservative, then $\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{r}\,'dt = \int_C \vec{F}\cdot d\vec{r}$

is path independent

because $\vec{F}\cdot\vec{T}\,ds = \vec{F}\cdot\vec{r}\,'dt$

$$= \underbrace{\left\langle \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle}_{\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} = \frac{d\phi}{dt}} dt$$

by chain rule

$$\begin{tikzcd} & \phi \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[d, dash] & & y \arrow[d, dash] \\ t & & t \end{tikzcd}$$

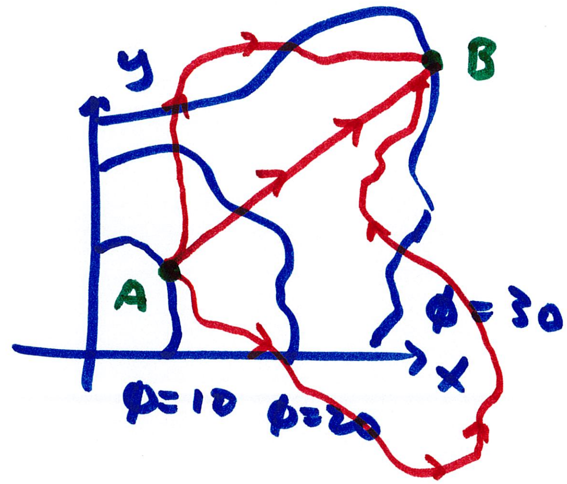
so, $\int_C \vec{F}\cdot\vec{r}\,'dt = \int_C \frac{d\phi}{dt}\,dt = \int_C d\phi = \phi(B) - \phi(A)$

end location start location

Fundamental Theorem of Line Integrals

$$\int_C \vec{F} \cdot \vec{r}\,' dt = \phi(B) - \phi(A)$$

if $\vec{F} = \vec{\nabla}\phi$



ALL of these paths give the same $\int_C \vec{F} \cdot \vec{r}\,' dt$

$= \phi(B) - \phi(A)$

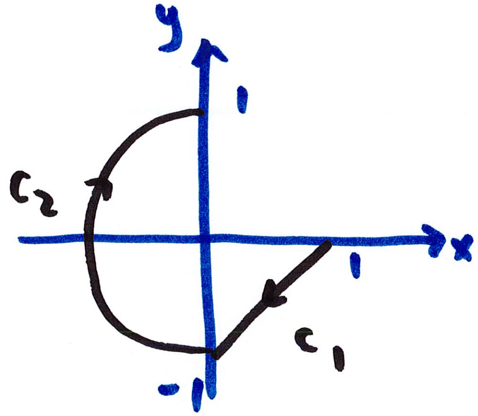
$= 30 - 10 = 20$

example

$$\int_C \vec{F}\cdot d\vec{r} \qquad \vec{F} = \langle x+y,\ x\rangle$$

C: line segment from (1, 0) to (0, -1)

then along the left half of $x^2+y^2=1$ to (0, 1)



let's first try as a regular line integral

parametrize: $C_1: \vec{r}(t) = \langle 1-t,\ t\rangle \quad 0\le t\le 1$

$C_2: \vec{r}(t) = \langle -\sin t,\ -\cos t\rangle \quad 0\le t\le \pi$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{r}\,'\,dt$$

$$= \underbrace{\int_0^1 \langle 1-2t,\ 1-t\rangle\cdot\langle -1,\ -1\rangle\,dt}_{C_1} + \underbrace{\int_0^{\pi} \langle -\sin t-\cos t,\ -\sin t\rangle\cdot\langle -\cos t,\ \sin t\rangle\,dt}_{C_2}$$

$$= \int_0^1 (3t-2)\,dt + \int_0^{\pi} (\sin t\cos t + \cos^2 t - \sin^2 t)\,dt = \dots = \boxed{-\frac{1}{2}}$$

not terribly bad, but we can use the Fundamental Theorem of Line Integrals to get the answer MUCH quicker

$\vec{F} = \langle x+y, x \rangle$

we know this is conservative from the earlier example and

$\phi = \frac{1}{2}x^2 + xy + c$

so, $\int_C \vec{F} \cdot \vec{r}\,' dt = \phi(B) - \phi(A)$

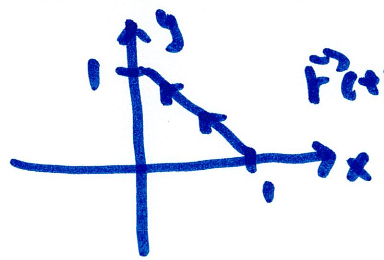
end location $(\underset{x}{0}, \underset{y}{1})$

start location $(\underset{x}{1}, \underset{y}{0})$

$$= \left[\frac{1}{2}(0)^2 + (0)(1) + c\right] - \left[\frac{1}{2}(1)^2 + (1)(0) + c\right] = \boxed{-\frac{1}{2}}$$

alternatively, since path doesn't matter, we could have chosen a simpler path w/ same start and end in the line integral

for example,



$\vec{r}(t) = \langle 1-t, t \rangle$