17.4 Green's Theorem

NOT on exam 2

let $\vec{F} = \langle f, g \rangle$ be a 2D vector field

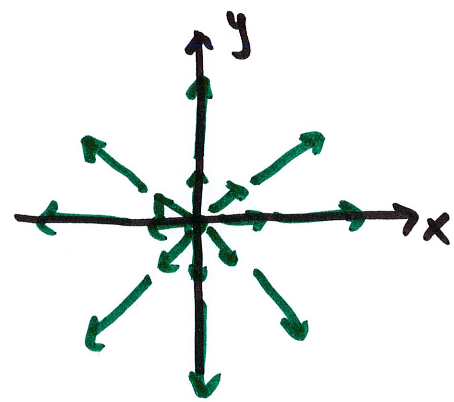
the quantity $\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\vec{k}$ or $\langle 0, 0, g_x - f_y \rangle$ is called the

<u>curl</u> of $\vec{F}$

written as $\text{curl}\,\vec{F}$

$|\text{curl}\,\vec{F}| = g_x - f_y$ is a measure of rotation in $\vec{F}$

for example, $\vec{F} = \langle x, y \rangle$
$f \quad g$

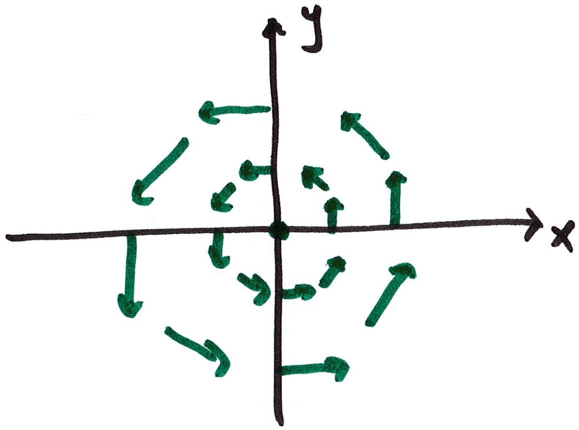


clearly has no rotation

the curl: $\text{curl}\,\vec{F} = \langle 0, 0, g_x - f_y \rangle = \langle 0, 0, 0 \rangle = \vec{0}$

$|\text{curl}\,\vec{F}| = 0 \rightarrow$ indicating no rotation

$\vec{F} = \langle -y, x \rangle$



there is a rotation

$\text{curl}\,\vec{F} = \langle 0, 0, g_x - f_y \rangle$

$= \langle 0, 0, 1-(-1) \rangle = \langle 0, 0, 2 \rangle$

$|\text{curl}\,\vec{F}| = 2 \neq 0$ indicating presence of rotation in $\vec{F}$

recall if $g_x - f_y = 0$ in $\vec{F} = \langle f, g \rangle$, then $\vec{F}$ is <u>conservative</u>

and since $|\text{curl}\,\vec{F}| = g_x - f_y$, this means a conservative vector field has <u>no rotation</u> $\rightarrow$ <u>irrotational</u>

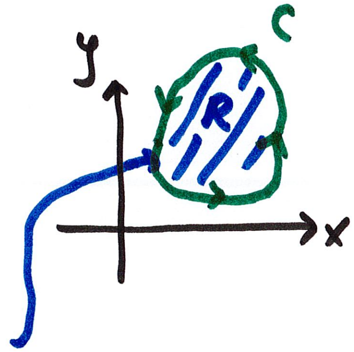
Green's Theorem

if $\vec{F} = \langle f, g \rangle$ and C is a simple closed path traversed in the counter clockwise direction, then

$$\oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

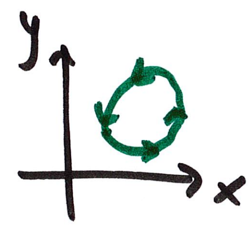
indicates that C is a closed loop

region enclosed by C



why is this true?

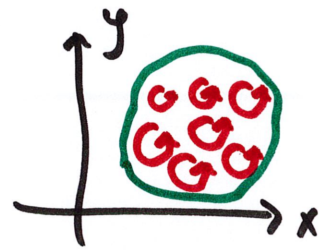
left side : $\oint_C f\,dx + g\,dy = \oint_C \vec{F} \cdot \vec{T}\,ds = \oint_C \vec{F} \cdot d\vec{r}$



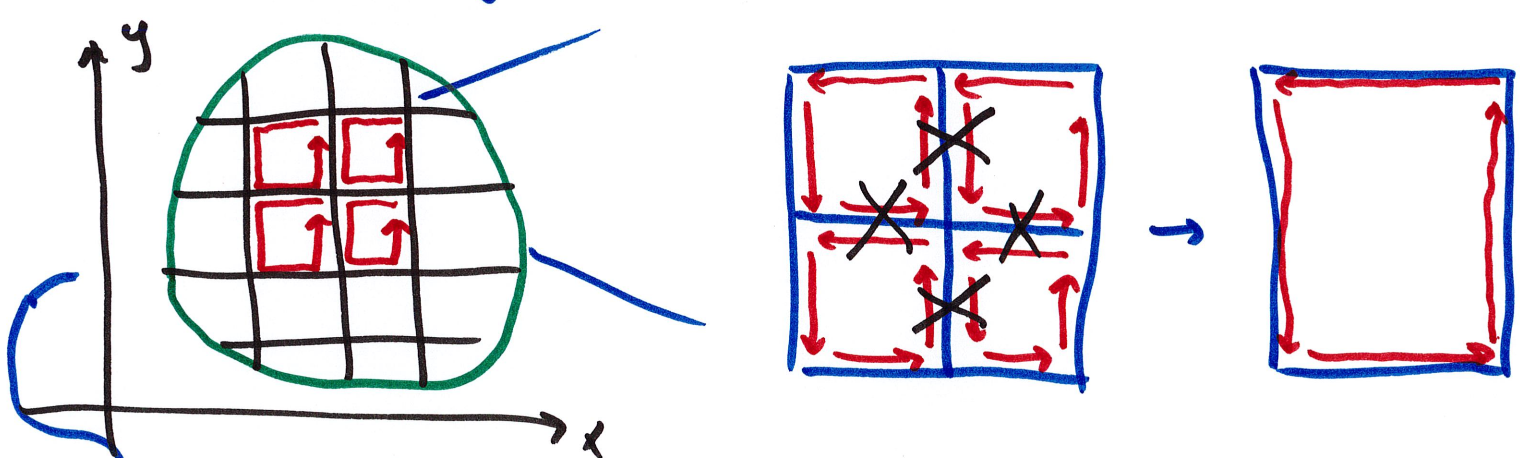
$$= \oint_C \langle f, g \rangle \cdot \langle dx, dy \rangle$$

this accumulates the vector field along the direction of travel along C → **Circulation** right side: $\iint_R \underbrace{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)}_{|\text{curl}\, \vec{F}|} dA$ (measures rotation)

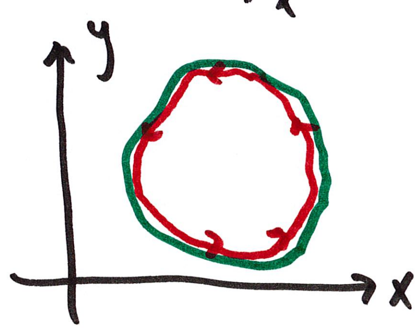
~~left~~ right side accumulates the micro rotations inside boundary



notice the inner things cancel out



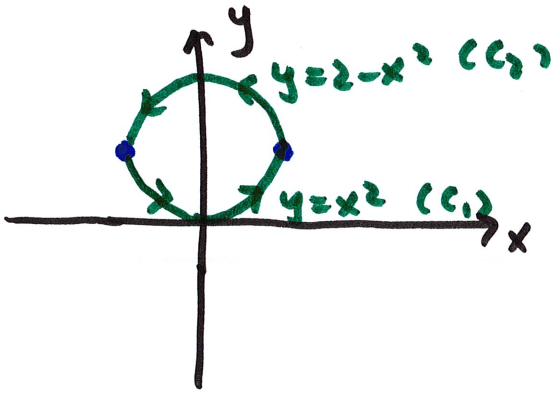
so,



only the flow along boundary matters

(which is what the left side of Green's Theorem does) example $\vec{F} = \langle y+2,\ x^2+1 \rangle$

C: $(-1,1)$ to $(1,1)$ along $y=x^2$, then back to $(-1,1)$ along $y = 2-x^2$



let's do this as a line integral first

C_1: $\vec{r} = \langle t, t^2 \rangle \quad -1 \le t \le 1$

C_2: $\vec{r} = \langle -t, 2-t^2 \rangle \quad -1 \le t \le 1$

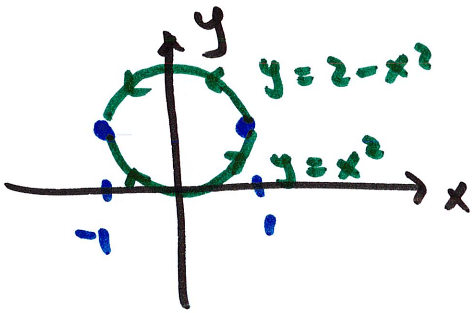
$$\oint_C \vec{F}\cdot\vec{r}\,'\,dt = \int_{-1}^{1} \underbrace{\langle t^2+2,\ t^2+1\rangle}_{\vec{F} \text{ using } x,y \text{ from } r \text{ on } C_1} \cdot \langle 1, 2t\rangle\,dt + \int_{-1}^{1} \langle 4-t^2,\ t^2+1\rangle \cdot \langle -1, -2t\rangle\,dt$$

$$= \int_{-1}^{1} (2t^3+t^2+2t+2)\,dt + \int_{-1}^{1} (-2t^3+t^2-2t-4)\,dt = \cdots = \frac{14}{3} - \frac{22}{3} = \boxed{-\frac{8}{3}}$$

not too bad, but Green's Theorem allows us to trade the line integral for a double integral

$$\oint_C f\,dx + g\,dy = \iint_R (g_x - f_y)\,dA$$

$$\vec{F} = \langle \underset{f}{y+2},\ \underset{g}{x^2+1} \rangle \qquad g_x - f_y = 2x - 1$$



$$-1 \le x \le 1$$

$$x^2 \le y \le 2 - x^2$$

$$\iint_R (g_x - f_y)\,dA = \int_{-1}^{1} \int_{x^2}^{2-x^2} (2x-1)\,dy\,dx$$

$$= \int_{-1}^{1} (2x-1)(2-2x^2)\,dx = \int_{-1}^{1} (-4x^3 + 2x^2 + 4x - 2)\,dx$$

$$= \cancel{\int_{-1}^{1} 4x - 2x^3 - 2x} + x^2 - \quad = \dots = \boxed{-\frac{8}{3}}$$

Green's Theorem also allows us to calculate the area of a region by doing a line integral along the boundary

$$\oint_C f\,dx + g\,dy = \iint_R (g_x - f_y)\,dA$$

if $g_x - f_y = 1$, the right side is $\iint_R dA$ = area

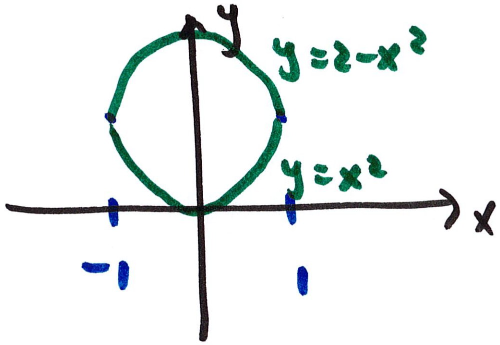
so, if we can come up with a $\vec{F} = \langle f, g \rangle$ such that $g_x - f_y = 1$, then the left side should also give us the area

there are many possibilities, one is $\vec{F} = \langle \underbrace{-\tfrac{1}{2}y}_{f}, \underbrace{\tfrac{1}{2}x}_{g} \rangle$

using that f, g on left side

$$\oint_C -\tfrac{1}{2}y\,dx + \tfrac{1}{2}x\,dy = \boxed{\tfrac{1}{2}\oint_C x\,dy - y\,dx} = \text{area enclosed by } C$$

example Area enclosed by C from the previous example



$C_1 : \vec{r} = \langle t, t^2 \rangle \quad -1 \le t \le 1$

$C_2 : \vec{r} = \langle -t, 2 - t^2 \rangle \quad -1 \le t \le 1$

$$\text{area enclosed} = \frac{1}{2} \oint_C x\,dy - y\,dx$$

$$= \underbrace{\frac{1}{2} \int_{-1}^{1} (\underset{x}{t})(\underset{dy}{2t\,dt}) - (\underset{y}{t^2})(\underset{dx}{dt})}_{C_1} + \underbrace{\frac{1}{2} \int_{-1}^{1} (\underset{x}{-t^2})(\underset{dy}{-2t\,dt}) - (\underset{y}{2-t^2})(\underset{dx}{-dt})}_{C_2}$$

$$= \frac{1}{2} \int_{-1}^{1} t^2\,dt + \frac{1}{2} \int_{-1}^{1} (t^2 + 2)\,dt = \frac{1}{3} + \frac{7}{3} = \boxed{\frac{8}{3}}$$

if we replace $\vec{T}$ with $\vec{N}$ (unit normal) in Green's Theorem

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R (g_x - f_y)\,dA$$

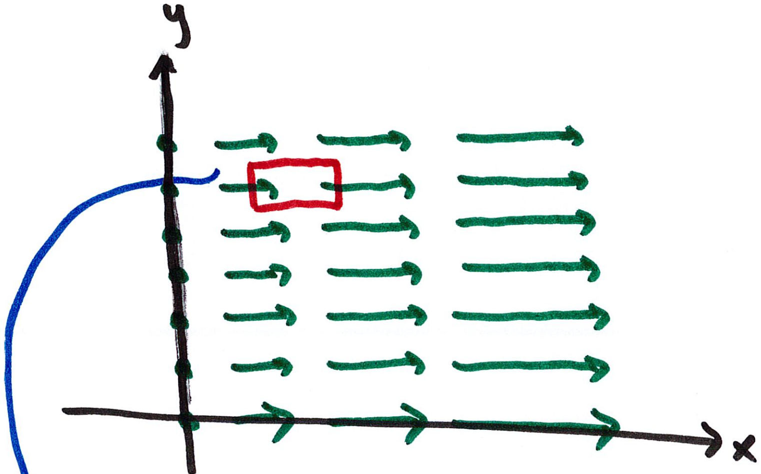
we end up getting :
$$\oint_C f\,dy - g\,dx = \iint_R (f_x + g_y)\,dA$$

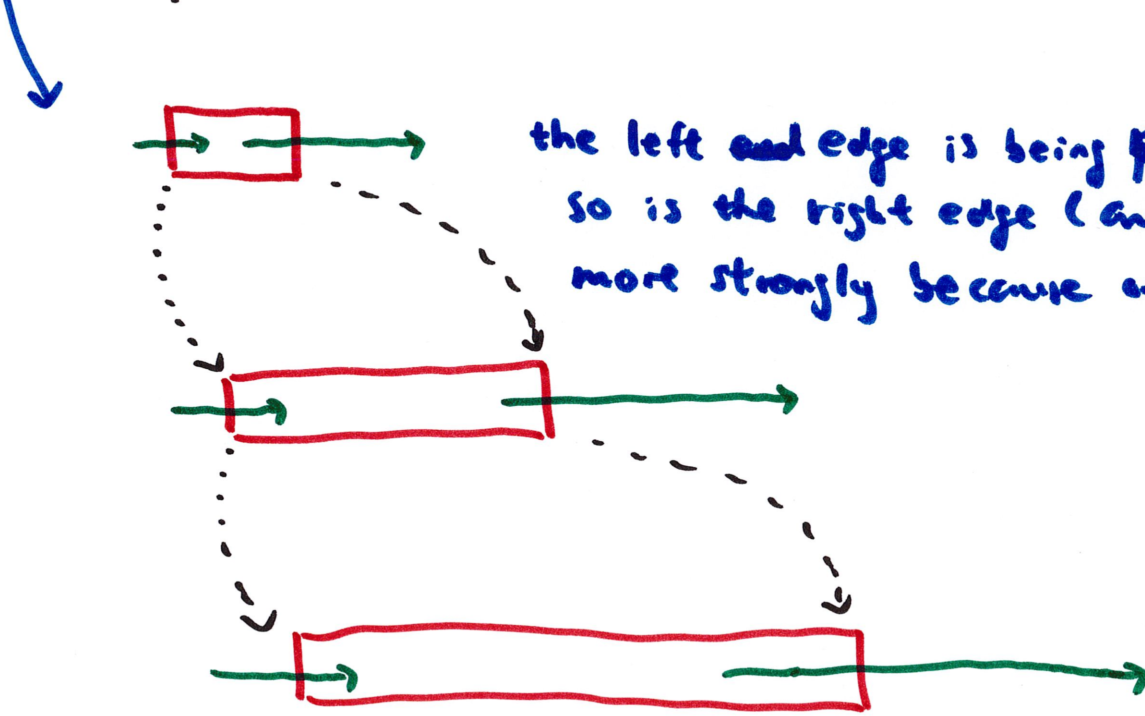
this then allows to calculate the flux integral as a double integral

the quantity $f_x + g_y$ is called the Divergence of $\vec{F}$, $\operatorname{div}\vec{F}$

it measures the change of a small volume as it flows along the vector field for example, $\vec{F} = \langle \underset{f}{x}, \underset{g}{0} \rangle \rightarrow \text{div}\,\vec{F} = f_x + g_y = 1$

this indicates an imaginary box flowing in the field would grow bigger over time





the left ~~and~~ edge is being pushed to the right but so is the right edge (and it is actually being pulled more strongly because of the longer vector)

So, the box gets bigger, consistent with the positive divergence