MA 261 Exam 2 Location

(Thursday, 4/8, 6:30 pm)

RECITATION TA	EXAM ROOM
LI	WTHR 200
HOGLE	WTHR 200
GRANADOS	EE 129
HIATT	EE 129
ENYEART	Hiler Theater (WALC)
HARDWICK	MTHW 210
GLENN SECTIONS 753, 761	RHPH 172
GLENN SECTION 769	EE 170
SMITH SECTIONS 777, 785	LILY G126
SMITH SECTION 793	EE 170

17.5 Curl and Divergence

(NOT on exam 2)

let's revisit the gradient: $\vec{\nabla} f = \underbrace{\langle f_x, f_y, f_z \rangle}_{\text{vector}}$

must be a vector-like thing

scalar

in fact, $\vec{\nabla}$ (the "del operator") is defined as

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial k}$$

in 2D: $\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}$

is a vector-like thing whose components have no meaning until operating on something

so, $\vec{\nabla} f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

Since $\vec{\nabla}$ behaves like a vector, we can do dot and cross products with it.

$$\boxed{\text{Curl } \vec{F} = \vec{\nabla} \times \vec{F}}$$

2D: $\vec{F} = \langle f, g, 0 \rangle$

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & 0 \end{vmatrix} = \left\langle \frac{\partial}{\partial y} 0 - \frac{\partial g}{\partial z}, -\frac{\partial}{\partial x} 0 + \frac{\partial f}{\partial z}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$

$$= \langle 0, 0, g_x - f_y \rangle$$

exactly how we defined curl for 2D vector field in previous section

Same process for 3D.

<u>example</u> $\vec{F} = \langle xy^2z^3,\ x^3yz^2,\ x^2y^3z \rangle$

$\operatorname{curl}\vec{F} = \vec{\nabla} \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^3 & x^3yz^2 & x^2y^3z \end{vmatrix}$$

$$= \Big\langle \frac{\partial}{\partial y}(x^2y^3z) - \frac{\partial}{\partial z}(x^3yz^2),\ -\Big(\frac{\partial}{\partial x}(x^2y^3z) - \frac{\partial}{\partial z}(xy^2z^3)\Big),$$
$$\frac{\partial}{\partial y}(xy^2z^3) - \frac{\partial}{\partial x}(x^3yz^2) \Big\rangle$$

$$= \cdots = \langle 3x^2y^2z - 2x^3yz,\ 3xy^2z^2 - 2xy^3z,\ 3x^2yz^2 - 2xyz^3 \rangle$$

we know that if $\vec{F}$ is conservative, then $\vec{F} = \vec{\nabla}\phi = \langle \phi_x, \phi_y, \phi_z \rangle$

let's find the curl of such a vector field

$$\text{curl}\,\vec{F} = \text{curl}(\text{grad}\,\phi) = \vec{\nabla} \times (\vec{\nabla}\phi) = \vec{\nabla} \times \langle \phi_x, \phi_y, \phi_z \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi_x & \phi_y & \phi_z \end{vmatrix} = \langle \underbrace{\phi_{zy} - \phi_{yz}}_{0}, \underbrace{-(\phi_{zx} - \phi_{xz})}_{0}, \underbrace{\phi_{yx} - \phi_{xy}}_{0} \rangle$$

$$= \langle 0, 0, 0 \rangle = \vec{0}$$

$\boxed{\text{so, the curl of a conservative vector field is } \vec{0}}$

this fact agrees with how we checked if a vector field is conservative in the previous section

in other words, a conservative vector field is irrotational if we take the dot product of del with a vector field we get the divergence

$$\boxed{\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}}$$

in 2D, $\vec{F} = \langle f, g, 0 \rangle$

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, 0 \rangle$$

$$= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

(this is how we defined the divergence for a 2D vector field)

Same idea for 3D vector field

<u>example</u> $\vec{F} = \langle xy^2z^3, x^3yz^2, x^2y^3z \rangle$

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}$$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle xy^2z^3, x^3yz^2, x^2y^3z \rangle$$

$$= \frac{\partial}{\partial x}(xy^2z^3) + \frac{\partial}{\partial y}(x^3yz^2) + \frac{\partial}{\partial z}(x^2y^3z)$$

$$= \boxed{y^2z^3 + x^3z^2 + x^2y^3}$$

don't forget : curl is a <u>vector</u>
divergence is a <u>scalar</u>

curl is easy to see in a vector field → look for rotation

what about the divergence ?

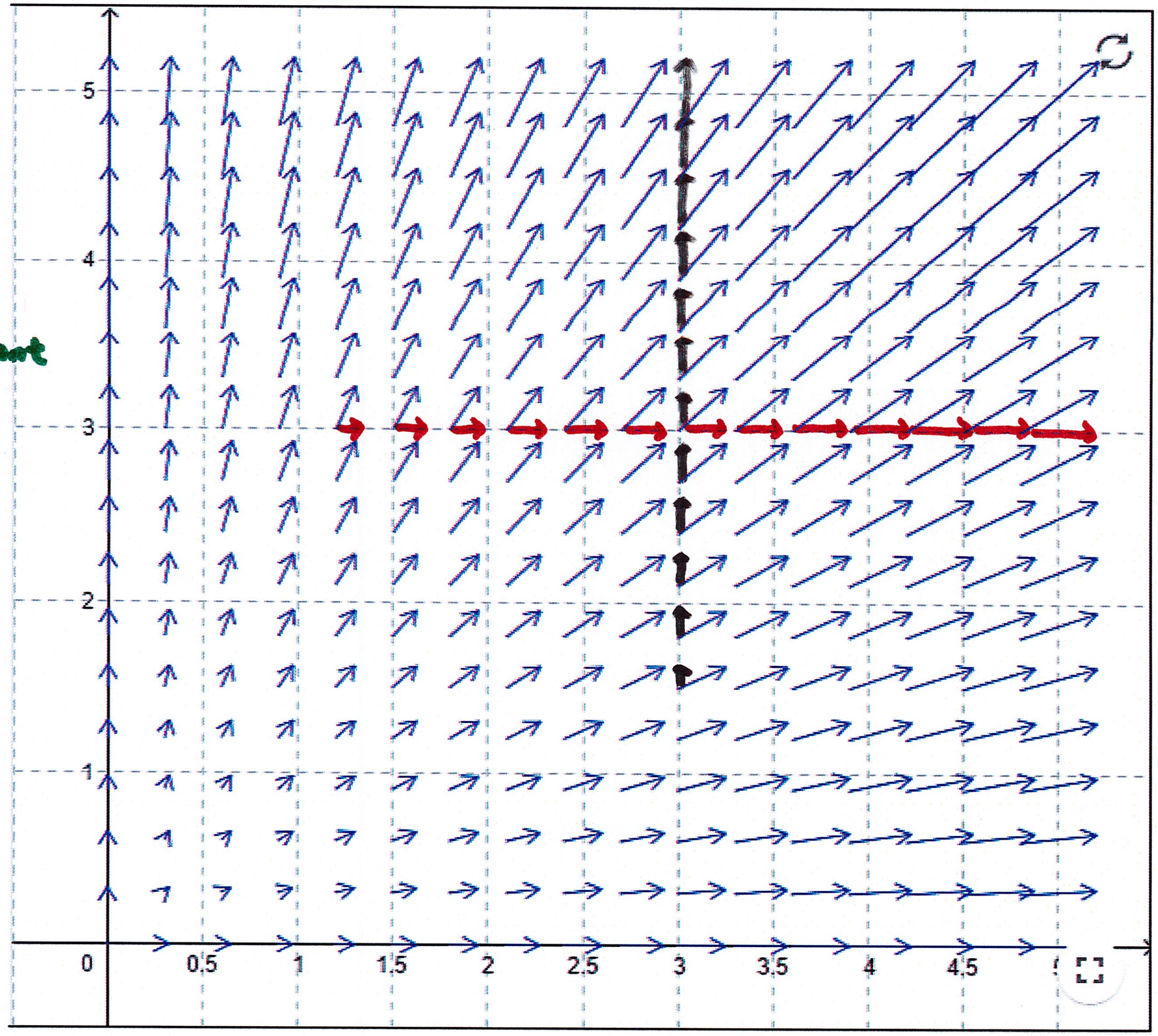
no rotation visible – no curl (must be conservative)

can we "see" the divergence?

$\vec{F} = \langle f, g \rangle$

x-compont of a vector

y-component



$\operatorname{div} \vec{F} = f_x + g_y$

how does the x-component change as x increases (as we move to the right) (the red arrows on the right)

this shows $f_x > 0$

$g_y > 0$ (see the black arrows)

so, we know $\operatorname{div} \vec{F} = f_x + g_y > 0$