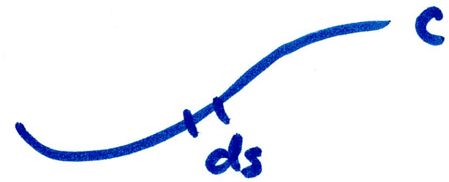
17.6 Surface Integrals (part 1)

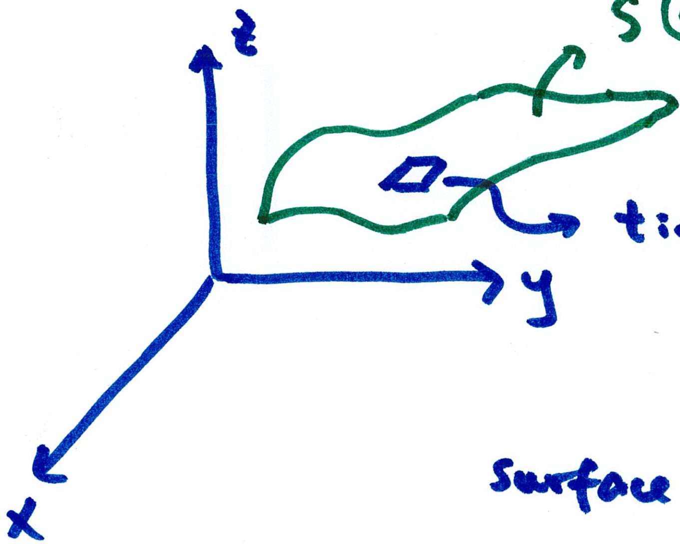
line integral : $\int_C f(x,y,z)\,ds$



accumulate $f(x,y,z)$ along curve C

Surface integral : accumulates something all over a surface

S (capital S)



tiny piece of the surface, call it dS

Surface integral looks like : $\iint_S f(x,y,z)\,dS$

we need to parametrize the surface, just like how we parametrized a curve

Curve C: $\vec{r}(t)$, $a \le t \le b$

Surface S: $\vec{r}(\underbrace{u, v}_{\text{two parameters}})$ u, v in some domain

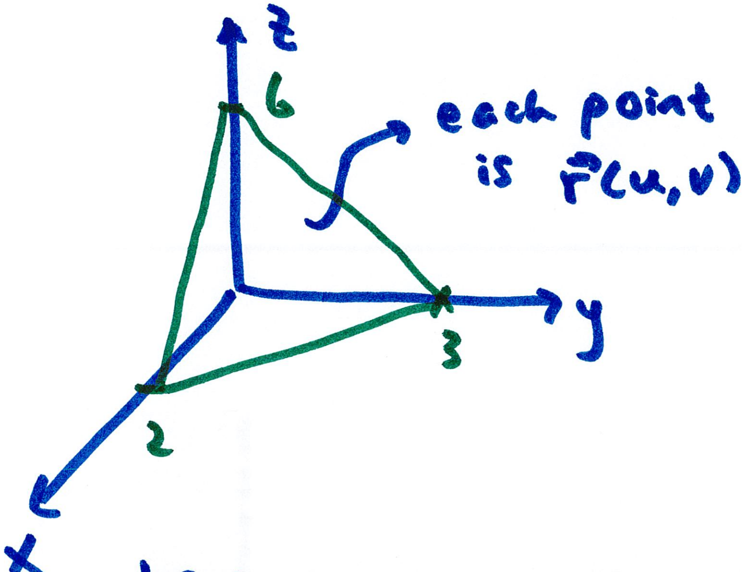
~~line~~ from $(-1, 1)$ to $(2, 4)$ along $y = x^2$

$\vec{r}(t) = \langle t, t^2 \rangle$ $-1 \le t \le 2$

↳ because $y = x^2$

the steps to parametrize a surface is nearly identical

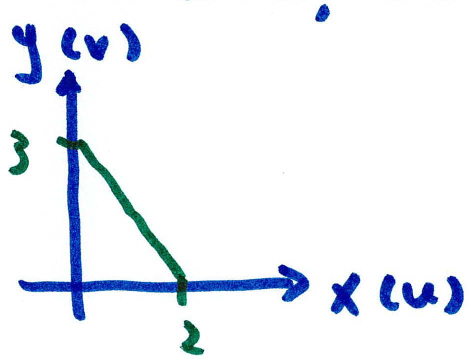
example Parametrize the part of the plane $3x+2y+z=6$ in the first octant.



goal: write $\vec{r}(u,v)$ to give location of each point

here is one possible parametrization

let $u=x$, $v=y$, then $z=6-3u-2v$

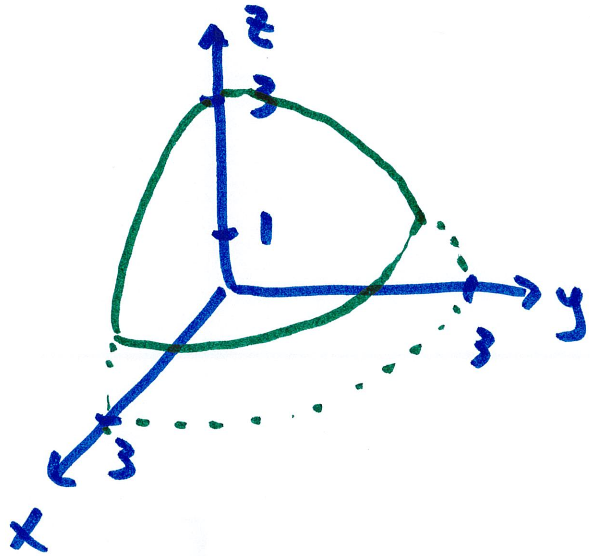


$$0 \le u \le 2$$
$$0 \le v \le 3-\frac{3}{2}u$$

so, this surface can be parametrized as

$$\boxed{\vec{r}(u,v) = \langle u,\ v,\ 6-3u-2v \rangle \qquad 0 \le u \le 2, \qquad 0 \le v \le 3-\tfrac{3}{2}u}$$

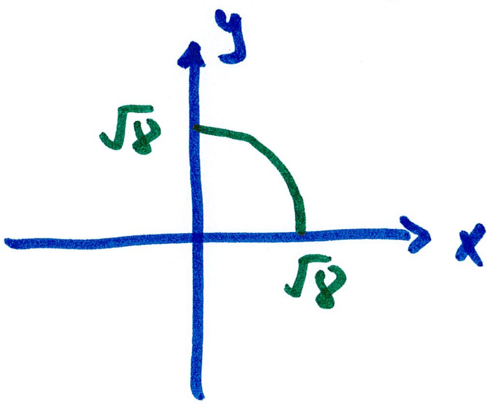
__example__ $x^2+y^2+z^2=9$ in first octant, $1 \le z \le 3$



one possible way to parametrize: use Cartesian

project onto xy-plane: at $z=1$, $x^2+y^2+1=9$

$x^2+y^2=8$



let $u=x$, $v=y$

then $z=\sqrt{9-u^2-v^2}$

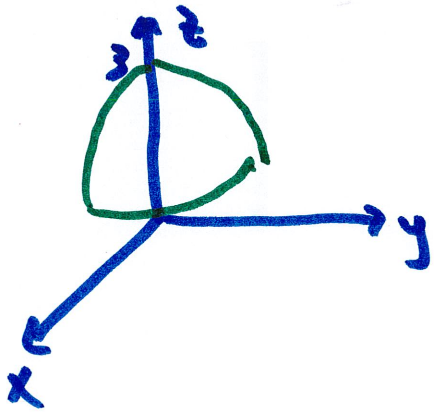
$0 \le u \le \sqrt{8}$

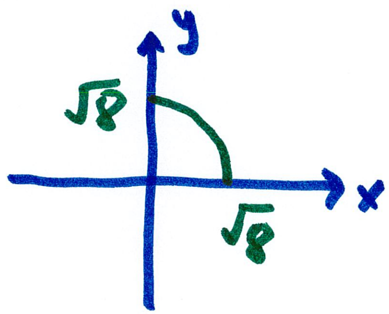
$0 \le v \le \sqrt{8-u^2}$

One possible parametrization:

$$\boxed{\vec{r}(u,v) = \left\langle u,\ v,\ \sqrt{9-u^2-v^2} \right\rangle \qquad 0 \le u \le \sqrt{8},\quad 0 \le v \le \sqrt{8-u^2}}$$

another parametrization: cylindrical





$0 \le r \le \sqrt{8}$

$0 \le \theta \le \pi/2$

$1 \le z \le \sqrt{9-r^2}$

let $u = r$, $\quad v = \theta$

then $1 \le z \le \sqrt{9-u^2}$

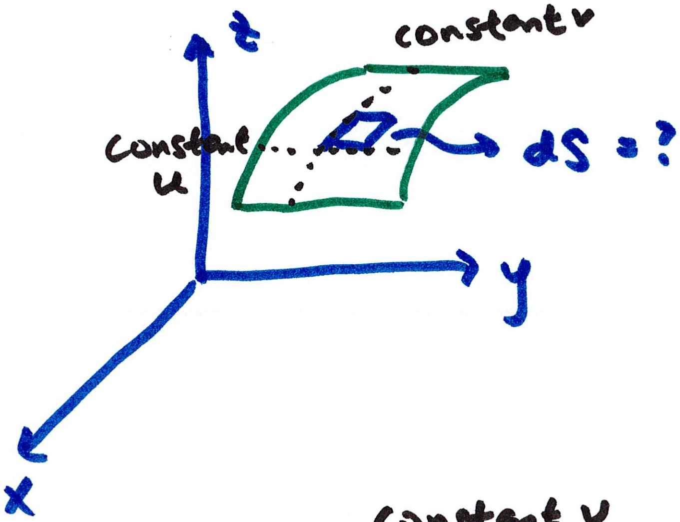
$$\boxed{\vec{r}(u,v) = \left\langle \underset{x = r\cos v}{u\cos v},\ \underset{y = r\sin v}{u\sin v},\ \sqrt{9-u^2} \right\rangle \qquad 0 \le u \le \sqrt{8},\quad 0 \le v \le \pi/2}$$

Surface integral: $\iint_S f(x,y,z)\,\underbrace{dS}_{\text{small piece of the surface}}$

~~we~~

in a line integral $\int_C f(x,y,z)\,ds$

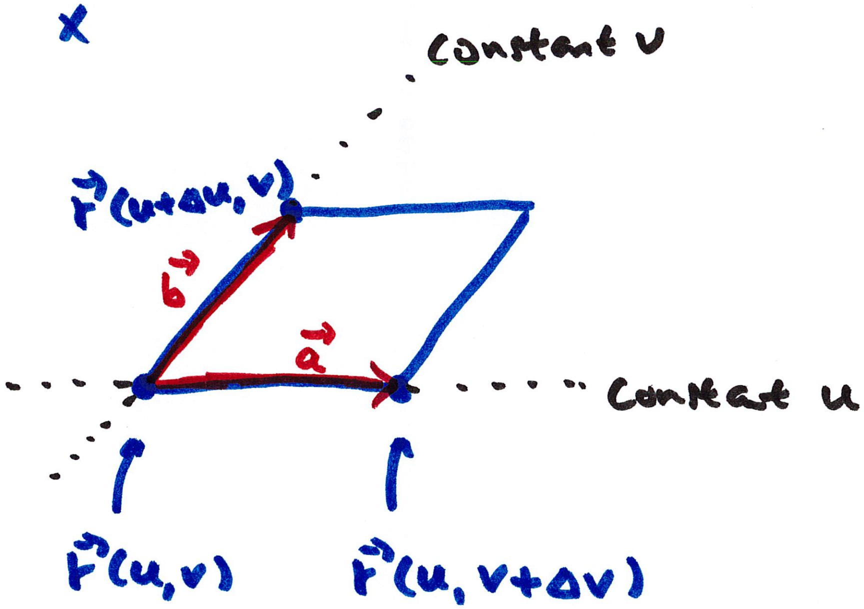
$ds = |\vec{r}\,'|\,dt$



dS : equivalent of ds for a surface

ds is small arc length

dS is a small area of the surface



goal: describe $\vec{a}$, $\vec{b}$ in terms of u, v, then area of dS is $|\vec{a} \times \vec{b}|$

recall $\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$

so, $\frac{\partial f}{\partial x} \cdot h \approx f(x+h, y) - f(x,y)$

or $\underline{f_x \cdot h \approx f(x+h, y) - f(x,y)}$

from last page: $\vec{a} = \vec{r}(u, v+\Delta v) - \vec{r}(u,v) \to \vec{a} = \vec{r}_v \Delta v$

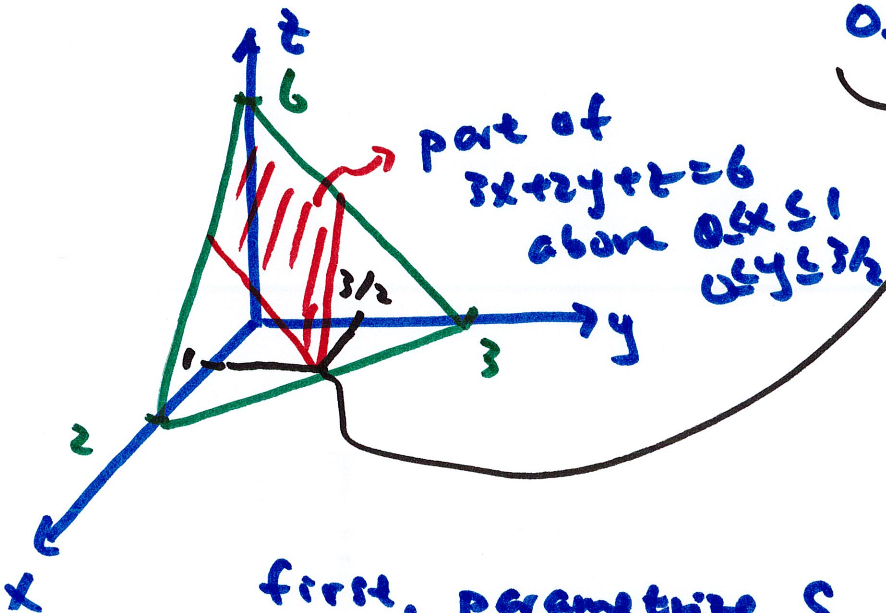
$\vec{b} = \vec{r}(u+\Delta u, v) - \vec{r}(u,v) \to \vec{b} = \vec{r}_u \Delta u$

so, $dS = |\vec{a} \times \vec{b}| = |\vec{r}_v \Delta v \times \vec{r}_u \Delta u|$

$= |\vec{r}_v \times \vec{r}_u| \Delta v \Delta u = |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$

then $\Delta u, \Delta v \to du, dv$, $\boxed{dS = |\vec{r}_u \times \vec{r}_v| \, du dv}$

so, surface integral: $\iint_S f \, dS$ (if $f=1$, this gives surface area) example $\iint_S (x+y)\,dS$ S: part of the plane $3x+2y+z=6$ in the first octant above $0 \le x \le 1$, $0 \le y \le 3/2$



first, parametrize S

let's use the parametrization from the earlier example

$\vec{r}(u,v) = \langle u, v, 6-3u-2v \rangle \qquad 0 \le u \le 1, \ 0 \le v \le 3/2$

$dS = |\vec{r}_u \times \vec{r}_v| \, du\,dv$

$\left.\begin{array}{l} \vec{r}_u = \dfrac{\partial \vec{r}}{\partial u} = \langle 1, 0, -3 \rangle \\ \vec{r}_v = \dfrac{\partial \vec{r}}{\partial v} = \langle 0, 1, -2 \rangle \end{array}\right\} \ |\vec{r}_u \times \vec{r}_v| = |\langle 3, 2, 1 \rangle| = \sqrt{14}$

$\boxed{dS = \sqrt{14}\, du\,dv}$

$$\iint_S (x+y)\,dS = \int_{v=0}^{3/2}\int_{u=0}^{1} (\underbrace{u}_{x \text{ in } \vec{r}(u,v)}+\underbrace{v}_{y \text{ in } \vec{r}(u,v)})\overbrace{\sqrt{14}\,du\,dv}^{dS}$$

$$= \cdots = \boxed{\frac{15\sqrt{14}}{8}}$$

could represent the mass of the red-shaded portion of the plane if density is $x+y$