17.6 Surface Integrals (part 2)

last time: $\iint_S g(x,y,z)\,dS = \underbrace{\iint_S g(u,v)\,|\vec{r}_u \times \vec{r}_v|\,dA}_{\text{do this for ANY coordinate system.}}$ ↗ dudv or dvdu

often, we get the surface: $z = f(x,y)$ z stated explicitly as function of x and y in Cartesian

we can come up with a convenient formula for that

$z = f(x,y)$

let $\vec{r}(u,v) = \vec{r}(x,y) = \langle x, y, f(x,y) \rangle$

$\vec{r}_u = \vec{r}_x = \langle 1, 0, f_x \rangle$

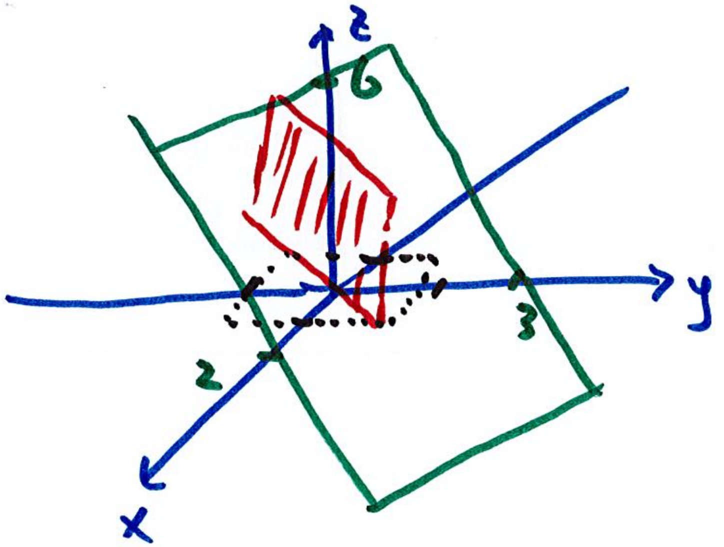
$\vec{r}_v = \vec{r}_y = \langle 0, 1, f_y \rangle$

$\vec{r}_u \times \vec{r}_v = \vec{r}_x \times \vec{r}_y = \langle -f_x, -f_y, 1 \rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{1 + f_x^2 + f_y^2}$ so, $dS = \sqrt{1+f_x^2+f_y^2}\,dA$ ↙ dxdy or dydx therefore, in the case where the surface is $z = f(x,y)$ in Cartesian, the surface integral is

$$\boxed{\iint_S g(x,y,z)\,dS = \iint_R g(x,y,z)\sqrt{1+f_x^2+f_y^2}\,dA}$$

<u>example</u> $\displaystyle\iint_S (x+y)\,dS$ S : the plane $\underbrace{z = 6-3x-2y}_{z = f(x,y) \text{ in Cartesian}}$ above the region $-1 \le x \le 1$, $-2 \le y \le 2$



$z = f(x,y) = 6-3x-2y$

$f_x = -3$

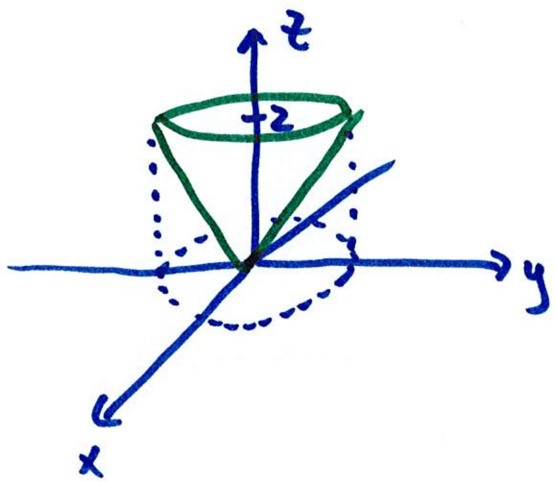
$f_y = -2$

$$dS = \sqrt{1+f_x^2+f_y^2}\,dydx = \sqrt{1+9+4}\,dydx$$

$$= \sqrt{14}\,dydx$$

$$\iint_S (x+y)\,dS = \int_{-1}^{1}\int_{-2}^{2}(x+y)\sqrt{14}\,dy\,dx = \cdots = \boxed{0}$$

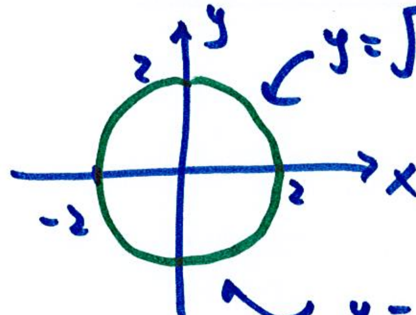
example Find the surface of $z^2 = x^2+y^2$, $0 \le z \le 2$



shadow on xy-plane is: $x^2+y^2=2^2$

circle radius 2

$z = f(x,y) = \sqrt{x^2+y^2}$



$y = \sqrt{4-x^2}$ $\quad -2 \le x \le 2$

$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

$y = -\sqrt{4-x^2}$

$$f_x = \frac{x}{\sqrt{x^2+y^2}} \qquad f_y = \frac{y}{\sqrt{x^2+y^2}}$$

so, $dS = \sqrt{1+f_x^2+f_y^2}\,dy\,dx = \sqrt{\dfrac{2(x^2+y^2)}{x^2+y^2}}\,dy\,dx = \sqrt{2}\,dy\,dx$

surface area: $\displaystyle\iint_S dS = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{2}\,dy\,dx$

$$= \sqrt{2} \underbrace{\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\,dx}$$

geometrically the area of circle radius 2 $= \pi(2)^2 = 4\pi$

$$= \boxed{4\sqrt{2}\pi}$$

alternative: evaluate in polar

$0 \le r \le 2$

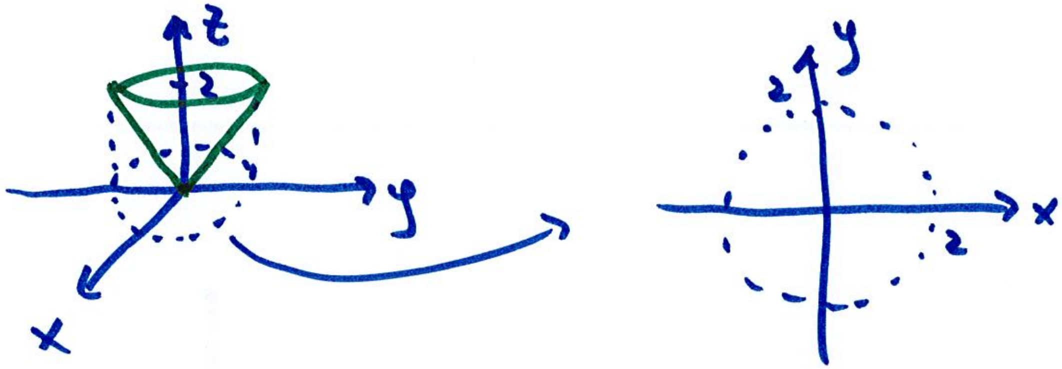
$0 \le \theta \le 2\pi$

$$\sqrt{2} \int_0^{2\pi} \int_0^2 \underbrace{r\,dr\,d\theta}_{dA = dy\,dx \text{ in polar}} = \dots = \boxed{4\sqrt{2}\pi}$$

if we parametrized in one coordinate system and then change to another, we must supply the missing pieces (e.g. r in $dA = r\,dr\,d\theta$)

but if we parametrized in ANY coordinate system, $|\vec{r}_u \times \vec{r}_v| \, dA$ ALWAYS takes care of whatever is needed.

example (same example) Surface area of $z^2 = x^2 + y^2$, $0 \le z \le 2$



let $u = r$, $v = \theta$

$0 \le u \le 2$

$0 \le v \le 2\pi$

$0 \le z \le \cancel{\sqrt{4-u^2}}$
u

$z = \sqrt{x^2+y^2} = \sqrt{r^2} = u$

$$\vec{r}(u,v) = \langle \overset{r\cos\theta}{u\cos v}, u\sin v, u \rangle$$

$$\left.\begin{array}{l} \vec{r}_u = \langle \cos v, \sin v, 1 \rangle \\ \vec{r}_v = \langle -u\sin v, u\cos v, 0 \rangle \end{array}\right\} \quad \begin{array}{l} \vec{r}_u \times \vec{r}_v = \langle -u\cos v, -u\sin v, u \rangle \\ |\vec{r}_u \times \vec{r}_v| = \sqrt{2}\, u \end{array}$$

$dS = |\vec{r_u} \times \vec{r_v}|\, dA = \sqrt{2}\, \underline{\underline{u}}\, du\, dv$

$\hookrightarrow$ $\underbrace{u = r \text{ in our parametrization}}$

what we needed to manually supply when changing from Cartesian to polar

BUT $|\vec{r_u} \times \vec{r_v}|$ ALWAYS does this automatically

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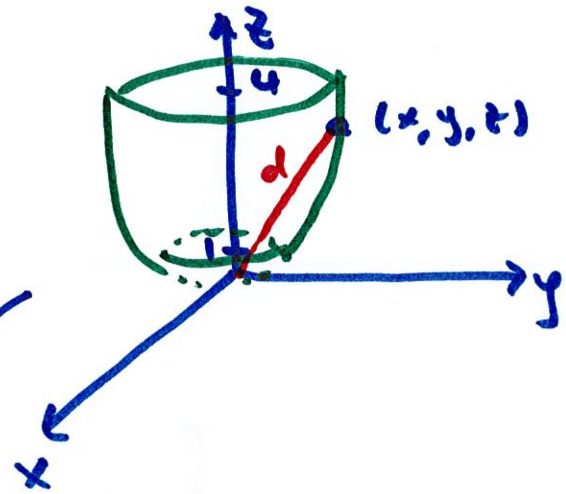
we can calculate the average value of $f(x,y,z)$ on S by taking similar process we used back when did double integrals

$\iint_S f(x,y,z)\, dS$ is the total accumulation of $f(x,y,z)$ over the surface

but if we replace $f(x,y,z)$ with its average value f_{avg}, then the total would be the same $\rightarrow$ $f_{avg} \iint_S dS$

$$\iint_S f(x,y,z)\, dS = f_{avg} \iint_S dS \quad \hookrightarrow \quad \boxed{f_{avg} = \frac{\iint_S f(x,y,z)\, dS}{\iint_S dS}}$$

example Find the average distance of the points on $z = x^2+y^2$ from the origin. $1 \le z \le 4$

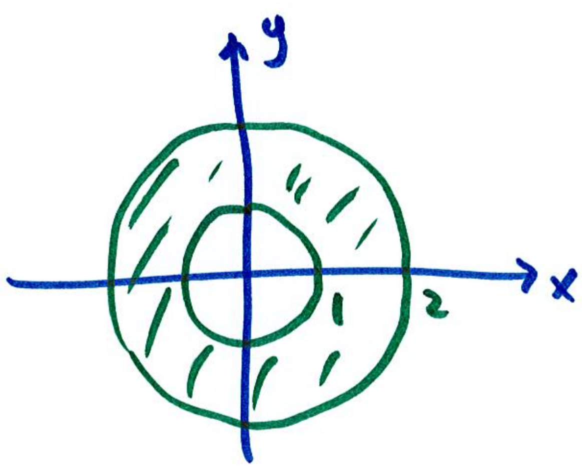


the thing ~~the~~ we wish to ~~average~~ average is the distance to the origin

$d = \sqrt{x^2+y^2+z^2}$ $\quad d_{avg} = ?$

~~cylin~~ cylindrical/polar is good

projection on xy-plane



let $u = r$, $v = \theta$

$1 \le u \le 2$ $\quad z = x^2+y^2 = r^2 = u^2$

$0 \le v \le 2\pi$

$\vec{r}(u,v) = \langle u\cos v, u\sin v, u^2 \rangle$

$\vec{r}_u = \langle \cos v, \sin v, 2u \rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 0 \rangle$

$$|\vec{r}_u \times \vec{r}_v| = \cdots = u\sqrt{1+4u^2}$$

$$f_{avg} = \frac{\iint_S f\,dS}{\iint_S dS}$$

$$\iint_S f\,dS = \int_0^{2\pi}\int_1^2 \sqrt{u^2+u^4}\cdot \underbrace{u\sqrt{1+4u^2}\,du\,dv}_{dS} = \cdots = 2\pi(15.21)$$

distance

$$d = \sqrt{x^2+y^2+z^2}$$
$$= \sqrt{r^2+(r^2)^2}$$
$$= \sqrt{r^2+r^4} = \sqrt{u^2+u^4}$$

$$\iint_S dS = \int_0^{2\pi}\int_1^2 \underbrace{u\sqrt{1+4u^2}\,du\,dv}_{dS} = 2\pi(4.91)$$

$$f_{avg} = \frac{2\pi(15.21)}{2\pi(4.91)} \approx 3.1$$