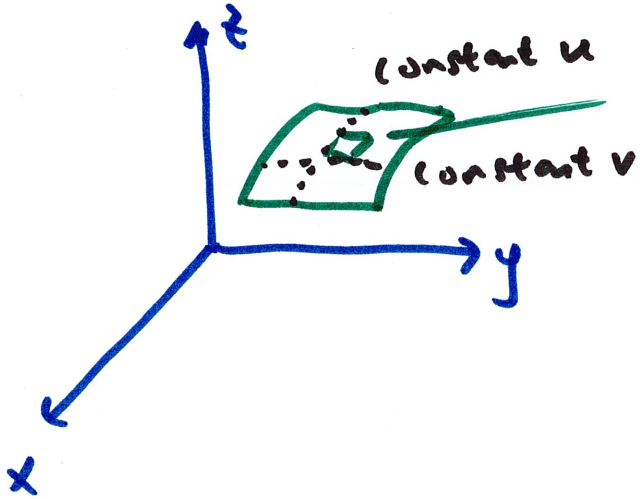
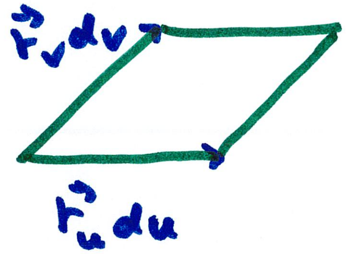
17.6 Surface Integrals (part 3)

last time: surface integrals in scalar field $\iint_S f(x,y,z)\underbrace{|\vec{r}_u \times \vec{r}_v|\,dA}_{}$

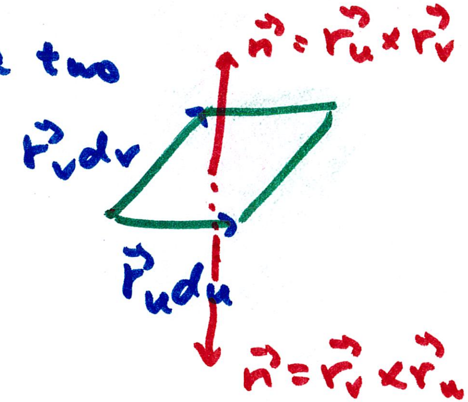
dS: area of small patch of S





$\vec{r}_u \times \vec{r}_v$ is the normal vector

but there are two



in scalar field surface integral, it didn't matter which normal we used $\rightarrow$ $|\vec{r}_u \times \vec{r}_v| = |\vec{r}_v \times \vec{r}_u|$ and $f(x,y,z)$ is scalar

but in vector field, it matters which we pick.

oriented surface : which normal do we work with?

by convention, we use the upward or outward pointing normal

→ we call the surface a positively-oriented surface

(others are ok, as long as we state it clearly)

surface integral in vector field : $\iint_S \vec{F} \cdot d\vec{S}$

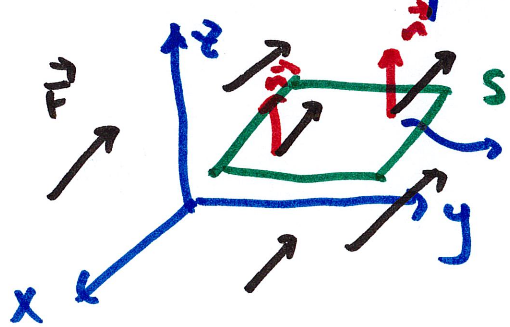
has an orientation

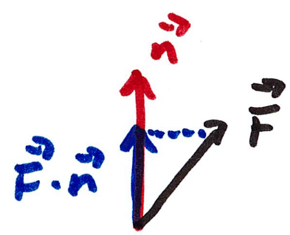
$$= \iint_S \vec{F} \cdot \vec{n}\, dS$$

unit normal

this is also called the Flux Integral

it accumulates the component of $\vec{F}$ in the same direction as $\vec{n}$





this accumulates the amount of "stuff" through the surface more useful form of $\iint\limits_S \vec{F}\cdot\vec{n}\,dS$

$dS \to |\vec{r_u}\times\vec{r_v}|\,dA$

unit normal $\dfrac{\vec{r_u}\times\vec{r_v}}{|\vec{r_u}\times\vec{r_v}|}$

$$= \iint\limits_S \vec{F}\cdot\frac{\vec{r_u}\times\vec{r_v}}{\cancel{|\vec{r_u}\times\vec{r_v}|}}\,\cancel{|\vec{r_u}\times\vec{r_v}|}\,dA = \boxed{\iint\limits_R \vec{F}\cdot\underbrace{(\vec{r_u}\times\vec{r_v})}\,dA}$$

or $\vec{r_v}\times\vec{r_u}$, whichever points in the positive orientation

alternative form for $z=z(x,y)$ and $\vec{F}=\langle f,g,h\rangle$

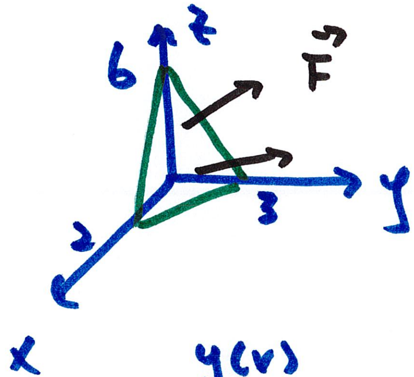
the above formula becomes $\iint\limits_R (-f\,z_x - g\,z_y + h)\,dA$

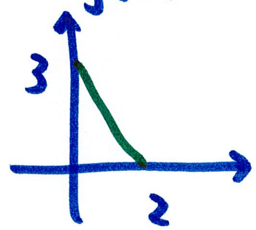
example $\vec{F} = \langle x, y, z \rangle$

S: plane $3x+2y+z=6$ in the first octant

normal is positive when pointing upward

↳ if not stated, assume up / out





first, parametrize the surface

let $u = x$, $v = y$, $z = 6 - 3u - 2v$

$0 \le u \le 2$

$0 \le v \le 3 - \frac{3}{2}u$

$\vec{r}(u,v) = \langle u, v, \cancel{3 - \frac{3}{2}u}\ 6 - 3u - 2v \rangle$

$\vec{r_u} = \langle 1, 0, -3 \rangle$

$\vec{r_v} = \langle 0, 1, -2 \rangle$

$\vec{r_u} \times \vec{r_v} = \langle 3, 2, 1 \rangle$ → is this pointing in the right direction? up?

yes, upward, because z-component > 0

if not, reverse the cross product

$$\vec{F} = \langle x, y, z \rangle = \langle u, v, 6 - 3u - 2v \rangle$$

$$\iint_S \vec{F} \cdot (\vec{r_u} \times \vec{r_v})\, dA = \underbrace{\int_0^2}_{u} \underbrace{\int_0^{3-\frac{3}{2}u}}_{v} \langle u, v, 6 - 3u - 2v \rangle \cdot \langle 3, 2, 1 \rangle \, dv\, du$$

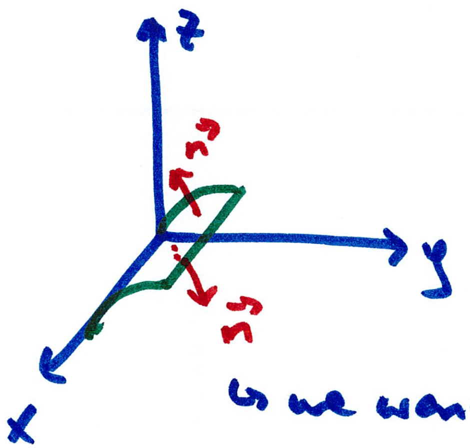
$$= \int_0^2 \int_0^{3-\frac{3}{2}u} 6 \, dv\, du = \cdots = \boxed{18}$$

this is some measure of the flow of $\vec{F}$ through the surface

example $\vec{F} = \langle -y, x, 1 \rangle$

S: cylinder $y = z^2$, $0 \le x \le 3$, $0 \le z \le 1$

normal is positive toward the positive y-axis



↳ we want this because it points toward positive y-axis

parametrize: $u = x$, $v = z$, $y = v^2$

$0 \le u \le 3$, $0 \le v \le 1$

$\vec{r}(u,v) = \langle u, v^2, v \rangle$

$\vec{r_u} = \langle 1, 0, 0 \rangle$

$\vec{r_v} = \langle 0, 2v, 1 \rangle$

$\vec{r_u} \times \vec{r_v} = \langle 0, -1, 2v \rangle$ points in the "positive" direction?

no, because y-component is < 0

so, the correct normal is $\vec{r_v} \times \vec{r_u} = \langle 0, 1, -2v \rangle$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_R \vec{F} \cdot (\vec{r_v} \times \vec{r_u})\, dA$$

$$= \underbrace{\int_0^3}_{u} \underbrace{\int_0^1}_{v} \underbrace{\langle -v^2, u, 1 \rangle}_{\vec{F} \text{ using the } x,y,z \text{ components of } \vec{r}(u,v)} \cdot \underbrace{\langle 0, 1, -2v \rangle}_{\vec{r_v} \times \vec{r_u}}\, dv\, du$$

$$= \cdots = \boxed{\frac{3}{2}}$$

→ positive means the net flow is with the normal (net flow is to the right)

<u>example</u> $\vec{F} = \dfrac{-\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}}$

S : sphere of radius a

normal pointing <u>outward</u>

parametrize S : in spherical $\rho = \text{constant} = a$

let $u = \phi$, $v = \theta$

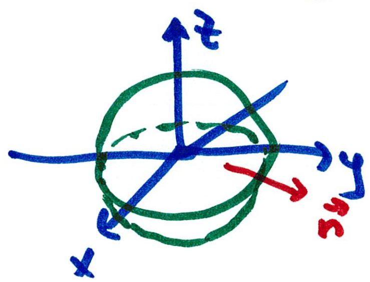
$0 \le u \le \pi$, $\quad 0 \le v \le 2\pi$

$$\vec{r}(u,v) = \langle \underbrace{a \sin u \cos v}_{\rho \sin\phi \cos\theta},\ a \sin u \sin v,\ a \cos u \rangle$$

$$\vec{r_u} = \langle a \cos u \cos v,\ a \cos u \sin v,\ -a \sin u \rangle$$

$$\vec{r_v} = \langle -a \sin u \sin v,\ a \sin u \cos v,\ 0 \rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle a^2 \sin^2 u \cos v,\ a^2 \sin^2 u \sin v,\ a^2 \cos u \sin u \rangle$$ pointing out?



in first octant, all component ≥ 0

↳ $\sin u, \cos v \ge 0$ so appears correct

check a few more to feel good about it

so that normal looks ok.

$$\vec{F} = \frac{-\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}}$$

$\vec{r}(u,v) = \langle a\sin u\cos v,$
$a\sin u\sin v,$
$a\cos u\rangle$

$$\iint_R \vec{F}\cdot(\vec{r_u}\times\vec{r_v})\,dA$$

$$= \int_{0}^{2\pi}\int_{0}^{\pi} \underbrace{\frac{-\langle a\sin u\cos v,\ a\sin u\sin v,\ a\cos u\rangle}{a^3}}_{\vec{F}} \cdot \langle \vec{r_u}\times\vec{r_v} \text{ from last page} \rangle \,du\,dv$$

v u

$= \dots = \boxed{-4\pi}$ negative, so net flow of $\vec{F}$ is against normal

(so, overall flow is into the sphere)