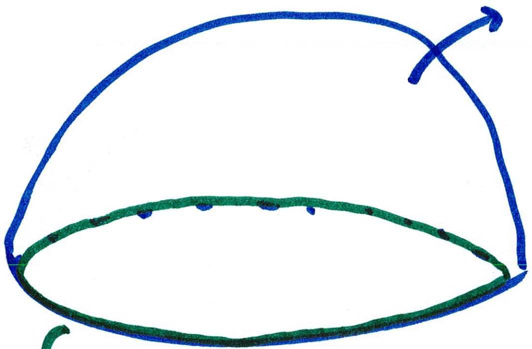
17.7 Stokes' Theorem (part 1)

Stokes' Theorem relates the surface integral of the curl of a vector field to a line integral along the boundary of the surface.

S (surface)



C

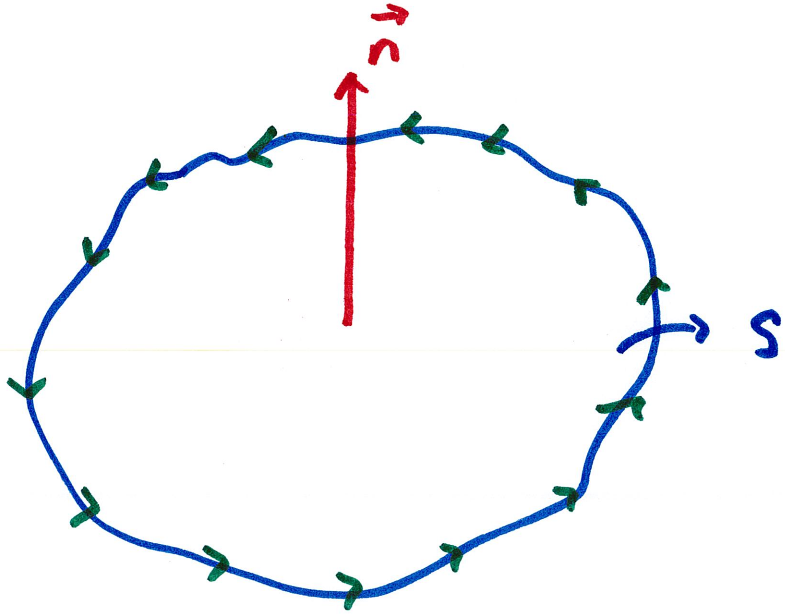
C (boundary)
the open part of the surface

opening is the boundary

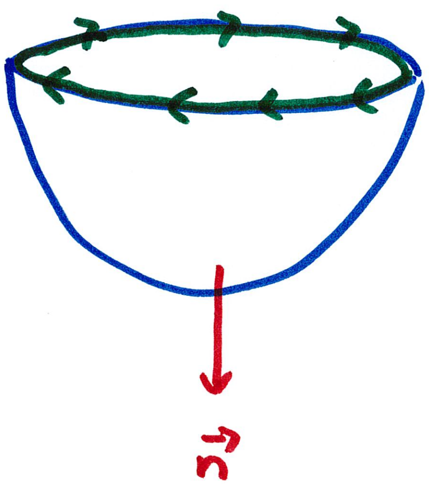
cup w/o lid

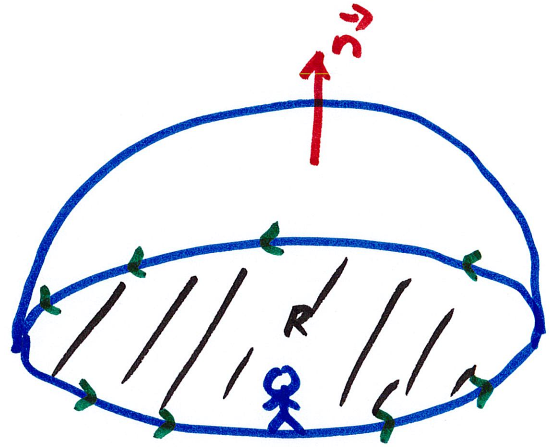
a closed surface (e.g. sphere) does NOT have a boundary

both the surface and the boundary curve are oriented and their orientations obey the right-hand rule



align thumb of right hand w/ the normal vector of surface the direction where our other fingers naturally curl is the direction of increasing t of the boundary curve.





imagine walking along boundary w/ head aligned with the normal vector, walk in direction such that area enclosed by boundary is on your LEFT

Stokes' Theorem

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

S : surface

C : boundary curve

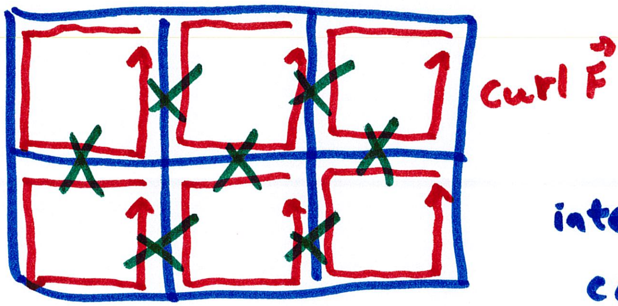
remember $d\vec{S} = n\,dS$

$= (\vec{r}_u \times \vec{r}_v)\,du\,dv$ or $(\vec{r}_v \times \vec{r}_u)\,du\,dv$

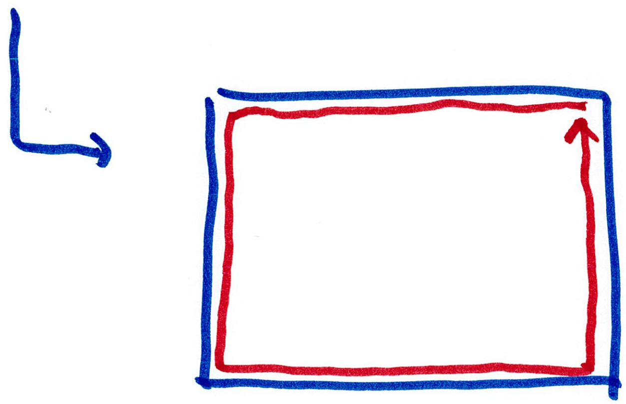
whichever points the normal in the correct direction

why is $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$?

the reason is the same as with Green's Theorem left side: $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ accumulates the curl of vector field (tiny whirlpools) all over the surface



interior "flows" cancel out



which is the same as accumulating the flow with the direction of the boundary curve on the boundary $\rightarrow$ $\oint_C \vec{F}\cdot d\vec{r}$

(right side)

Stokes' is a more general version of Green's Theorem

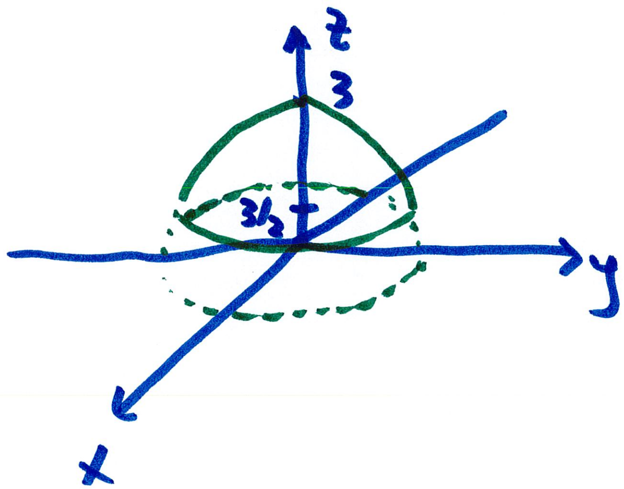
Stokes': surface can be flat or curved

Green's Theorem: surface is flat on xy-plane example $\vec{F} = \langle y, -x, 0 \rangle$

S: sphere radius 3, $z \geq 3/2$, normal is outward

let's verify Stokes' Theorem: $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$

left side:



parametrize S: use spherical for this sphere

$\rho = 3$ is fixed, so don't let it be a parameter (u or v)

let $u = \phi$, $v = \theta$

$\vec{r}(u,v) = \langle 3\sin u \cos v,\ 3 \sin u \sin v,\ 3\cos u \rangle$

$0 \leq v \leq 2\pi$

for u: $z \geq 3/2$

$3\cos u \geq 3/2$

$\cos u \geq 1/2 \rightarrow 0 \leq u \leq \pi/3$

$$\vec{r}_u = \langle 3\cos u \cos v,\ 3\cos u \sin v,\ -3\sin u \rangle$$

$$\vec{r}_v = \langle -3\sin u \sin v,\ 3\sin u \cos v,\ 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 9\sin^2 u \cos v,\ 9\sin^2 u \sin v,\ 9\sin u \cos u \rangle$$

is this "outward"?

yes.

(if not, flip it)

now do $\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S}$ $\quad \vec{F} = \langle y, -x, 0 \rangle \quad \operatorname{curl}\vec{F} = \cdots = \langle 0, 0, -2 \rangle$

$$= \iint_S \operatorname{curl}\vec{F} \cdot \vec{n}\, dS = \iint_S \operatorname{curl}\vec{F} \cdot \underbrace{(\vec{r}_u \times \vec{r}_v)}_{\text{or } \vec{r}_v \times \vec{r}_u \text{ whichever points correctly}}\, dA$$

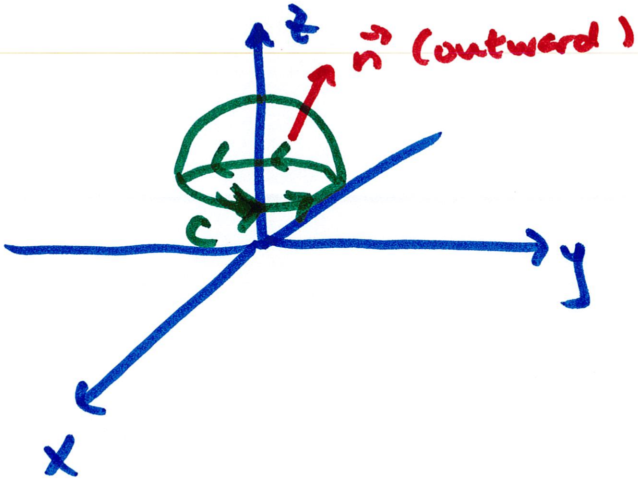
$$= \int_{0}^{2\pi}\!\!\int_{0}^{\pi/3} \underbrace{\langle 0, 0, -2 \rangle}_{\operatorname{curl}\vec{F}} \cdot \underbrace{\langle 9\sin^2 u \cos v,\ 9\sin^2 u \sin v,\ 9\sin u \cos u \rangle}_{\vec{r}_u \times \vec{r}_v}\, \underbrace{du\, dv}_{dA}$$

$v \quad u$

$$= \int_0^{2\pi}\!\!\int_0^{\pi/3} -18 \sin u \cos u \, du\, dv = \cdots = \boxed{-27\pi/2}$$

now let's see the right side: $\oint_C \vec{F}\cdot d\vec{r}$

first, parametrize C



since $\vec{n}$ is out, C is oriented counterclockwise when viewed from above (w/ z-axis pointing at us)

size of circle?

$x^2+y^2+z^2=9$ at $z=3/2$

$x^2+y^2+\frac{9}{4}=9$

$x^2+y^2=\frac{27}{4}$ C : circle radius $\frac{\sqrt{27}}{2}$

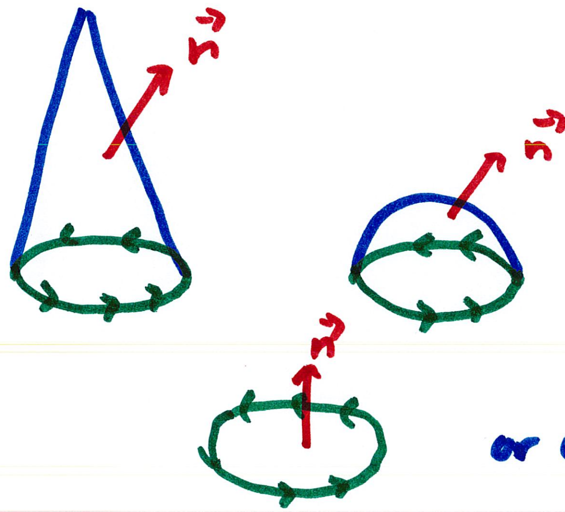
$\vec{r}(t)=\left\langle \frac{\sqrt{27}}{2}\cos t, \frac{\sqrt{27}}{2}\sin t, 3/2 \right\rangle$ $0\le t\le 2\pi$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \underbrace{\left\langle \frac{\sqrt{27}}{2}\sin t, -\frac{\sqrt{27}}{2}\cos t, 0\right\rangle}_{\vec{F}} \cdot \underbrace{\left\langle -\frac{\sqrt{27}}{2}\sin t, \frac{\sqrt{27}}{2}\cos t, 0\right\rangle dt}_{d\vec{r}=\vec{r}\,'dt}$$

$$=\int_0^{2\pi} -\frac{27}{4}\sin^2 t-\frac{27}{4}\cos^2 t \; dt = \int_0^{2\pi} -\frac{27}{4}\,dt = -\frac{27}{4}\cdot 2\pi = \boxed{-\frac{27}{2}\pi}$$

Stokes' Theorem : $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$

this means two different surfaces with the same boundary curve have the same value for $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$



same C ! so same $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$

or even a flat surface w/ same boundary!

can replace S w/ an easier one as long as C is the same