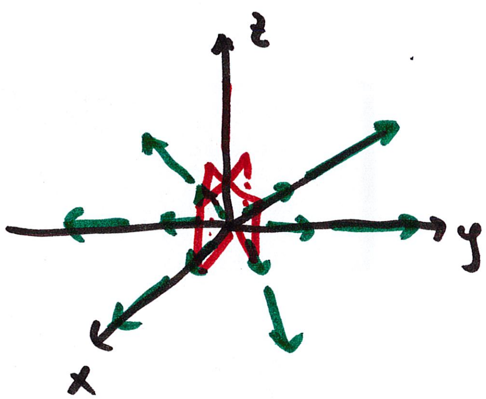
17.7 Stokes' Theorem (part 2)

what is the physical meaning of $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$?

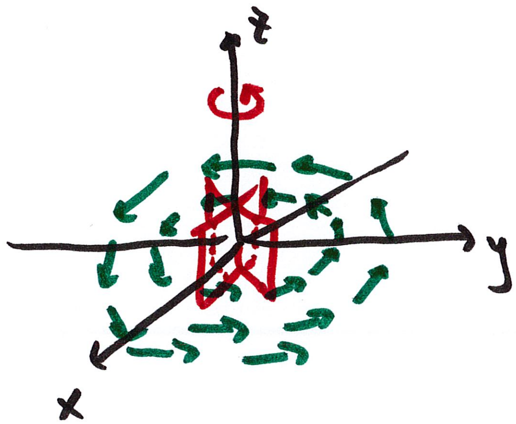
let's take a closer look at curl $\vec{F}$

$\vec{F} = \langle x, y, 0 \rangle$



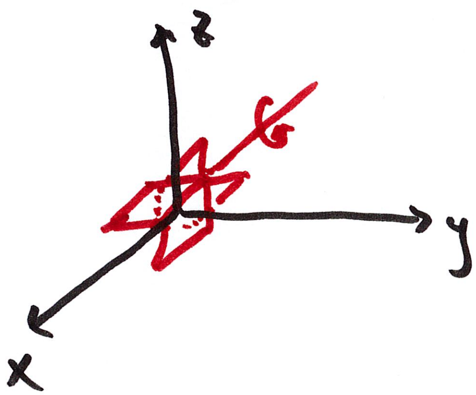
if a paddle wheel with axis along z-axis is placed at the origin

we don't expect the wheel to spin since the vector field pushes all fins radially out now look at $\vec{F} = \langle -y, x, 0 \rangle$

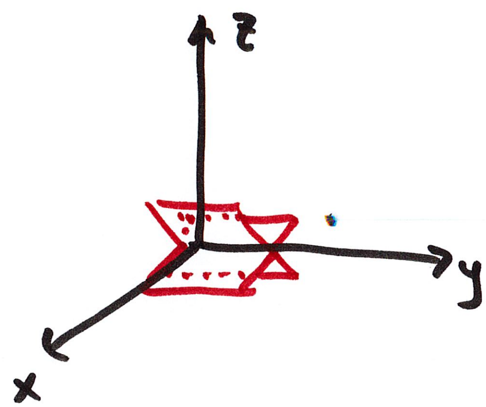


now expect the paddle wheel to spin if placed at the origin with axis along z-axis.

if we change how the axis is aligned, the spin changes



still spins, but not as fast/strong



no spin expected

first field: $\vec{F} = \langle x, y, 0 \rangle$ $\quad$ curl $\vec{F} = \vec{\nabla} \times \vec{F} = \langle 0, 0, 0 \rangle = \vec{0}$

second field: $\vec{F} = \langle -y, x, 0 \rangle$ $\quad$ curl $\vec{F} = \langle 0, 0, 2 \rangle$

this tells us we get the most spin from the vector field by placing the axis of the paddle wheel along the z-axis

if placed along x or y axis (0 in the curl), we get no spin

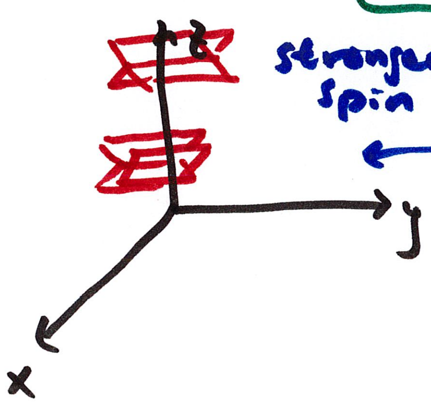
$\vec{F} = \langle 5 - z^2, 0, 0 \rangle$

curl $\vec{F} = \langle 0, -2z, 0 \rangle$

stick wheel w/ axis along y-axis to get spin from the vector field

and the higher $|z|$ is, the stronger the spin

the minus sign tells us the spin is opposite to what the right-hand rule dictates



stronger spin than

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_S \underbrace{\text{curl}\,\vec{F}\cdot\vec{n}\,dS}$$

accumulating the spin of a paddle wheel with its axis along the normal vector all over the surface

<u>example</u> S: upper half of $z^2 = a^2(1-x^2-y^2)$ $\quad$ a is some positive number

$\vec{n}$ is upward

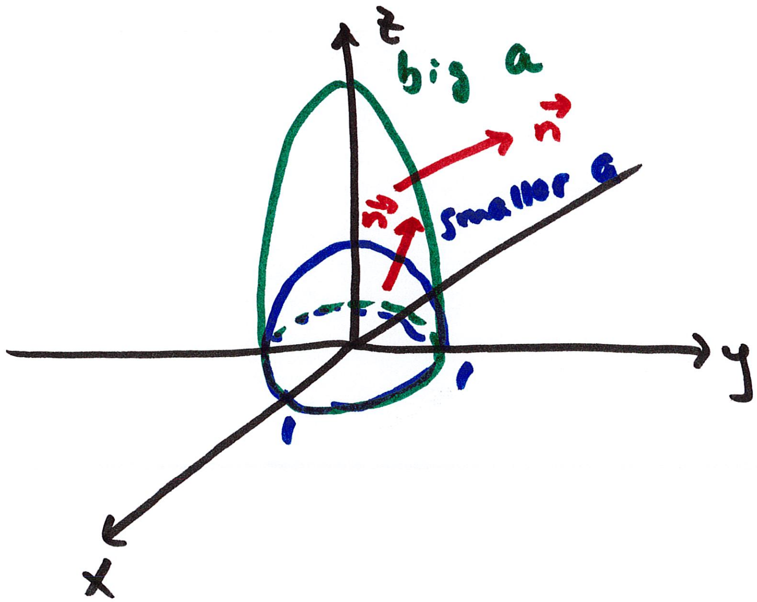
$\vec{F} = \langle x-y,\ y+z,\ z-x\rangle$

find a such that $\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS$ is maximized

shape? ellipsoid

$$z^2 = a^2(1-x^2-y^2)$$

$$\frac{z^2}{a^2} = 1-x^2-y^2 \rightarrow \quad x^2+y^2+\frac{z^2}{a^2} = 1$$



as a changes, $\vec{n}$ changes orientation

$\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS$ is clearly affected by a (amount of total spin with the paddle wheel axis along $\vec{n}$)

Stokes' Theorem: $\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS = \oint_C \vec{F}\cdot d\vec{r}$

this is suggesting that a has no effect because C, the boundary of the surface, is on xy-plane, and a does not change that

$x^2+y^2+\frac{z^2}{a^2}=1 \to$ x, y intercepts are always ± 1

the circle that intersects the xy-plane always has radius of 1 verify:

left side $\iint\limits_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS = \iint\limits_S \text{curl}\,\vec{F}\cdot\underbrace{(\vec{r_u}\times\vec{r_v})}_{\text{upward}}\,dA$

Surface: $x^2+y^2+\frac{z^2}{a^2}=1$

$z^2=a^2(1-x^2-y^2) \rightarrow z=a\sqrt{1-x^2-y^2}$

parametrize with cylindrical

let $u=r$, $v=\theta$ $\quad x=u\cos v \quad y=u\sin v$

$z=a\sqrt{1-u^2}$

$\vec{r}(u,v)=\langle u\cos v, u\sin v, a\sqrt{1-u^2}\rangle \quad 0\le u\le 1$

$\vec{r_u}=\langle \cos v, \sin v, \frac{1}{2}a(1-u^2)^{-1/2}(-2u)\rangle \quad 0\le v\le 2\pi$

$=\langle \cos v, \sin v, \frac{-au}{\sqrt{1-u^2}}\rangle$

$\vec{r_v}=\langle -u\sin v, u\cos v, 0\rangle$

$$\vec{r}_u \times \vec{r}_v = \left\langle \frac{a u^2 \cos v}{\sqrt{1-u^2}}, \frac{a u^2 \sin v}{\sqrt{1-u^2}}, u \right\rangle$$

upward? yes, since $0 \le u \le 1$

$$\vec{F} = \langle x-y,\ y+z,\ z-x \rangle$$

$$\text{curl } \vec{F} = \langle 1, 1, 1 \rangle$$

$$\iint_S \text{curl } \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

$$= \int_0^{2\pi} \int_0^1 \langle 1,1,1 \rangle \cdot \langle \quad \rangle \, du\, dv = \cdots = \boxed{\pi}$$

no a in this!

right side: $\oint_C \vec{F} \cdot d\vec{r}$

C: circle radius 1

$$\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle \quad 0 \le t \le 1$$

$$\vec{F} = \langle \cos t - \sin t,\ \sin t,\ -\cos t \rangle \qquad d\vec{r} = \langle -\sin t, \cos t, 0 \rangle\, dt$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-\cos t \sin t + \sin^2 t + \cos t \sin t)\, dt = \int_0^{2\pi} \sin^2 t\, dt$$

$$= \int_0^{2\pi} \frac{1}{2}(1 - \cos 2t)\, dt = \frac{1}{2} \cdot 2\pi = \boxed{\pi}$$

Stokes' Theorem $\iint_S \underline{\text{curl}\,\vec{F}} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$

is only applicable for the flux of the <u>curl</u> of the vector field

DO <u>NOT</u> use it for "regular" surface integral $\iint_S \vec{F} \cdot d\vec{S}$

NOT curl