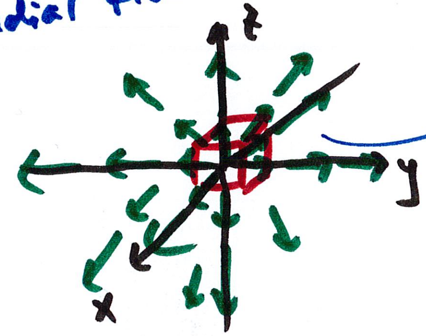
17.8 The Divergence Theorem (part 1)

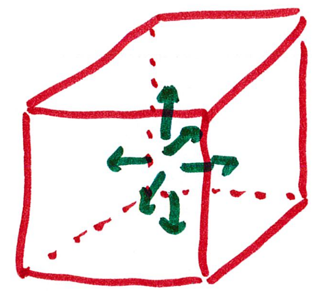
let's revisit the divergence : $\operatorname{div}\vec{F} = \vec{\nabla}\cdot\langle f, g, h\rangle = f_x + g_y + h_z$

$\vec{F} = \langle x, y, z\rangle \qquad \operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$

outward radial flow



→ tiny box at origin

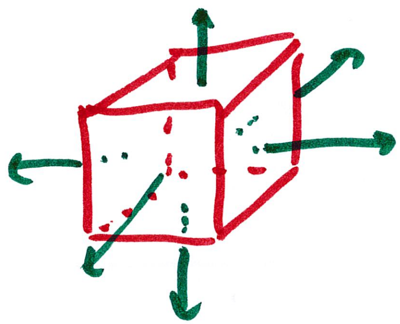


it's like air is pumped in and flowing outward from the origin

if the box is rigid, then the flow inside will expand box and make it bigger

$\rightarrow \operatorname{div}\vec{F} > 0$

if $\operatorname{div}\vec{F} < 0$ volume will shrink now imagine the box is not rigid and each face is made of a porous material so ~~the~~ air flows through



volume no longer changes but air flows through each face

↓

flux integral

so, the volume is related to flux integral

and that is what the <u>Divergence Theorem</u> says:

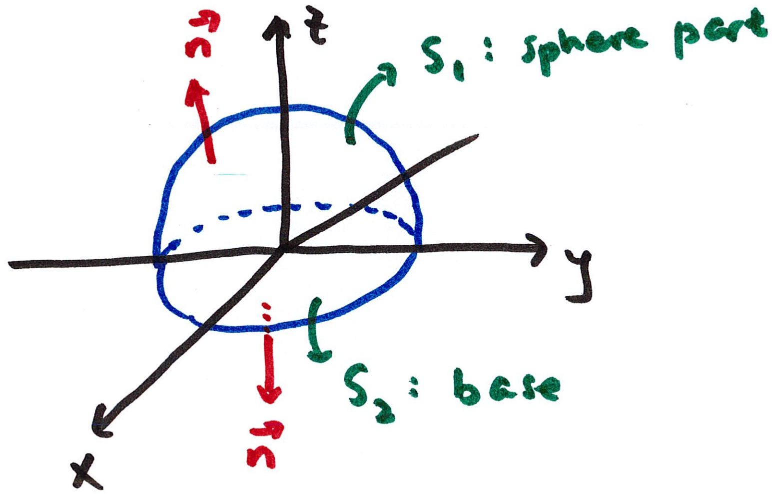
outward normal

$$\underbrace{\iint_S \vec{F}\cdot\vec{n}\,d\hat{S}}_{\text{flux integral through all surfaces}} = \underbrace{\iiint_D \operatorname{div}\vec{F}\,dV}_{\text{accumulation of divergence inside the space enclosed by the surface}}$$

this Theorem is primarily applied to <u>closed</u> surfaces example $\vec{F} = \langle x, y, z \rangle$

S : upper half of sphere radius 3 including the circular base at $z=0$

as usual, $\vec{n}$ is outward



note $\vec{n}$ outward is actually downward on S_2

let's verify Divergence Theorem : $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \operatorname{div}\vec{F}\,dV$

left side

S_1 : $\vec{r}(u,v) = \langle 3\sin u\cos v,\ 3\sin u\sin v,\ 3\cos u \rangle \qquad 0 \le u \le \pi/2\ ,\ 0 \le v \le 2\pi$

$\phi \rightarrow \quad \nwarrow \theta$

$\vec{r}_u = \langle 3\cos u\cos v,\ 3\cos u\sin v,\ -3\sin u \rangle$

$\vec{r}_v = \langle -3\sin u\sin v,\ 3\sin u\cos v,\ 0 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 9\sin^2 u\cos v,\ 9\sin^2 u\sin v,\ 9\cos u\sin v \rangle$ direction correct? yes

$S_2: \; \vec{r}(u,v) = \langle \underset{r}{u}\cos v, \; u\sin \underset{\theta}{v}, \; 0\rangle \qquad 0 \le u \le 3, \quad 0 \le v \le 2\pi$

$\vec{r}_u = \langle \cos v, \sin v, 0\rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 0\rangle$

$\vec{r}_u \times \vec{r}_v = \langle 0, 0, u\rangle$ direction correct?

no, since this is always upward, but we want downward

so we want $\vec{r}_v \times \vec{r}_u = -\langle 0,0,u\rangle = \langle 0, 0, -u\rangle$

now do $\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_S \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA$ on both surfaces

or $\vec{r}_v\times\vec{r}_u$

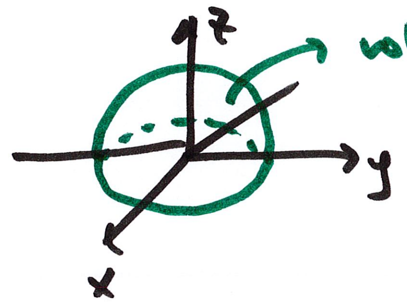
$$\underbrace{\int_0^{2\pi}\int_0^{\pi/2} \underbrace{\langle 3\sin u\cos v, 3\sin u\sin v, 3\cos u\rangle}_{\vec{F}\text{ using } x,y,z \text{ of } \vec{r}(u,v) \text{ on } S_1} \cdot \underbrace{\langle 9\sin^2 u\cos v, 9\sin^2 u\sin v, 9\cos u\sin u\rangle}_{\vec{r}_u\times\vec{r}_v}\,du\,dv}_{S_1}$$

$$+ \underbrace{\int_0^{2\pi}\int_0^{3} \underbrace{\langle u\cos v, u\sin v, 0\rangle}_{\vec{F}} \cdot \underbrace{\langle 0, 0, -u\rangle}_{\vec{r}_v\times\vec{r}_u}\,du\,dv}_{S_2} = \int_0^{2\pi}\int_0^{\pi/2} (27\sin^3 u + 27\cos^2 u\sin u)\,du\,dv$$

$$= \int_0^{2\pi}\int_0^{\pi/2} 27\sin u\,du\,dv = \boxed{54\pi}$$

the Divergence Theorem says it's equal to $\iiint_D \operatorname{div}\vec{F}\,dV$

$\vec{F} = \langle x, y, z \rangle \qquad \operatorname{div}\vec{F} = 3$



volume enclosed is D

$$\iiint_D \operatorname{div}\vec{F}\,dV = \iiint_D 3\,dV$$

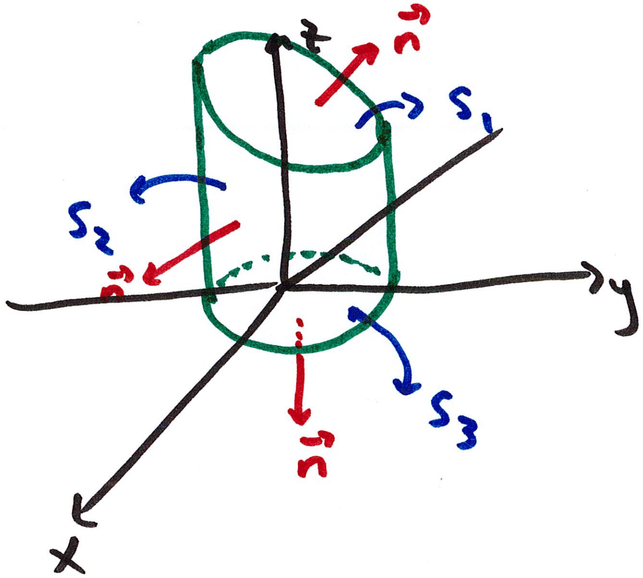
$$= 3\iiint_D dV$$

$$= 3\cdot\underbrace{\frac{1}{2}\cdot\frac{4}{3}\pi(3)^3}_{\text{half of volume of sphere radius 3}} = 2\cdot\pi(27) = \boxed{54\pi}$$

example $\vec{F} = \langle -y^3z^2,\ x^4,\ 4xy^2 \rangle$

S: surface of solid bounded by $\underbrace{x^2+y^2=1}_{\text{cylinder}}$, $\underbrace{z=10-y}_{\text{plane}}$, $\underbrace{z=0}_{\text{plane}}$

normal is outward



3 surfaces, 3 normals to track

looks easier to do the volume integral instead

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \operatorname{div}\vec{F}\,dV$$

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(-y^3z^2) + \frac{\partial}{\partial y}(x^4) + \frac{\partial}{\partial z}(4xy^2) = 0$$

so, $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \operatorname{div}\vec{F}\,dV = \iiint_D 0\,dV = 0$