17.8 The Divergence Theorem (part 2)

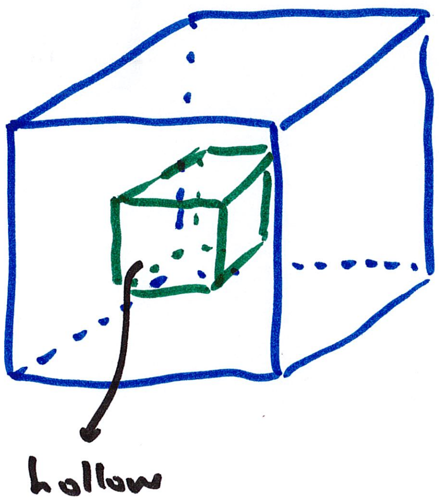
$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \operatorname{div}\vec{F}\,dV$$

D: space enclosed by S

$\vec{n}$ is assumed to be outward-pointing

if $\vec{n}$ is inward-pointing, then the flux is negative

what happens if S contains a hollow space in the middle,
for example a hollow sphere or cube?



S: the surface that forms the hollow cube
contains the outside and the inside

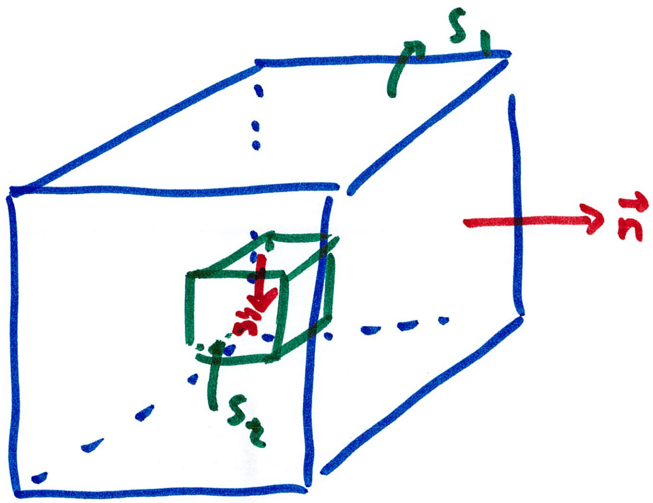
S_1: outer surface

S_2: inner surface

D: volume enclosed by S
↳ space between cubes

normal vector, as usual, is outward-pointing

↳ pointing away from interior of enclosed space



inside: pointing away means inward

so, for this situation, $\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_1} \vec{F}\cdot\vec{n}\,dS - \iint_{S_2} \vec{F}\cdot\vec{n}\,dS$

due to S_2 (inner cube) having inward normal

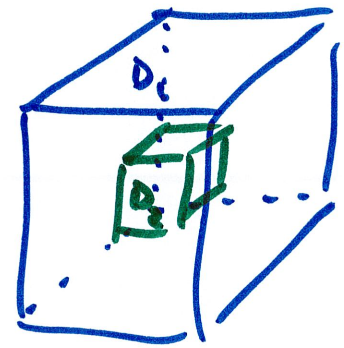
by Divergence Theorem, we see

$$\iint_{S_1} \vec{F}\cdot\vec{n}\,dS - \iint_{S_2} \vec{F}\cdot\vec{n}\,dS = \iiint_D \operatorname{div}\vec{F}\,dV$$

← space between curves

Furthermore, if D_2 is space contained by inner cube and D_2 is space contained by outer cube

then, applying Divergence Theorem again, we get



$$\iint_{S_1} \vec{F}\cdot\vec{n}\,dS - \iint_{S_2} \vec{F}\cdot\vec{n}\,dS = \iiint_{D} \operatorname{div}\vec{F}\,dV$$

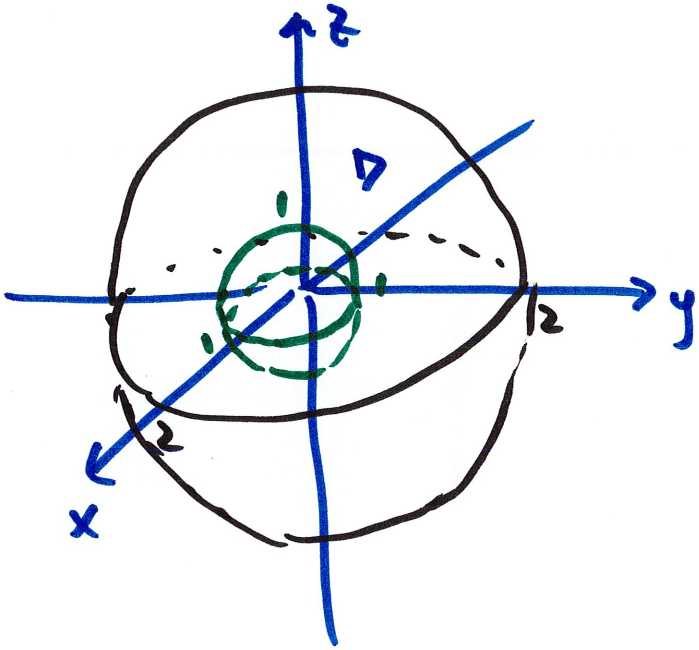
$$\iiint_{D_1} \operatorname{div}\vec{F}\,dV - \iiint_{D_2} \operatorname{div}\vec{F}\,dV = \iiint_{D} \operatorname{div}\vec{F}\,dV$$

in words, this means we calculate the flux out of the outer surface, then <u>subtract</u> the flux contributed by the inner cube (hollow portion)

<u>example</u> $\vec{F} = \langle x, y, z \rangle$ $\qquad$ $\operatorname{div}\vec{F} = 3$

D: between spheres of radii 2 and 1

normal is outward (away from enclosed space)



Divergence Theorem: $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \operatorname{div}\vec{F}\,dV$

between spheres

parametrize D: $\quad 1 \le \rho \le 2$

$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

so, the flux is $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_1^{2} \underbrace{3}_{\operatorname{div}\vec{F}}\ \underbrace{\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}_{dV}$

$$= 3\int_0^{2\pi}\int_0^{\pi} \tfrac{1}{3}\rho^3\Big|_1^2 \sin\phi\, d\phi\, d\theta = 7\int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta$$

$$= 7\cdot 2\pi\,(-\cos\phi)\Big|_0^{\pi} = 14\pi\,(1 - -1) = \boxed{28\pi}$$

we can also do $\underset{D_1}{\iiint} \operatorname{div} \vec{F}\, dV - \underset{D_2}{\iiint} \operatorname{div} \vec{F}\, dV$

$D_1 \to$ outer sphere

$D_2 \to$ inner sphere

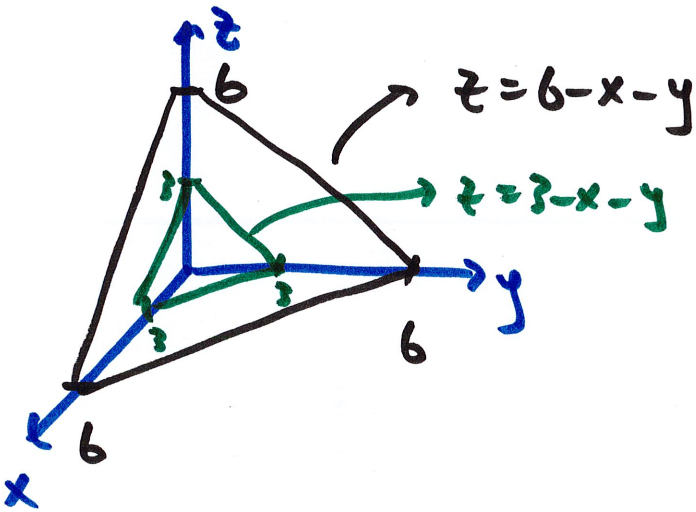
$$= \underset{D_1}{\iiint} 3\, dV - \underset{D_2}{\iiint} 3\, dV$$

$$= 3 \underbrace{\underset{D_1}{\iiint} dV}_{\text{volume of outer sphere}} - 3 \underbrace{\underset{D_2}{\iiint} dV}_{\text{volume of inner sphere}} = 3 \cdot \frac{4}{3}\pi (2)^3 - 3 \cdot \frac{4}{3} \pi (1)^3$$

$$= 32\pi - 4\pi = \boxed{28\pi}$$

example $\vec{F} = \langle x^2, -y^2, z^2 \rangle$

D: bounded by $z = 6 - x - y$ and $z = 3 - x - y$ in first octant



D: space between

flux integral

choices: 1) Surface integral for the surface enclosing D
(messy, five surfaces to parameterize)

2) $\iiint_D \operatorname{div} \vec{F} \, dV$

easiest → 3) $\iiint_{D_1} \operatorname{div} \vec{F} \, dV - \iiint_{D_2} \operatorname{div} \vec{F} \, dV$

D_1: contained by outer plane

D_2: contain by inner plane

D_1 :

$$0 \le x \le 6$$
$$0 \le y \le 6-x$$
$$0 \le z \le 6-x-y$$

D_2 :

$$0 \le x \le 3$$
$$0 \le y \le 3-x$$
$$0 \le z \le 3-x-y$$

$$\operatorname{div} \vec{F} = 2(x-y+z)$$

$$\int_0^6 \int_0^{6-x} \int_0^{6-x-y} 2(x-y+z)\, dz\, dy\, dx - \int_0^3 \int_0^{3-x} \int_0^{3-x-y} 2(x-y+z)\, dz\, dy\, dx$$

$$= \cdots = 108 - \frac{27}{4} = \boxed{\frac{405}{4}}$$

$\operatorname{div}\vec{F} = \frac{2}{\rho}$ in spherical

parametrize space between the spheres

$$\varepsilon \le \rho \le 1$$
$$0 \le \phi \le \pi$$
$$0 \le \theta \le 2\pi$$

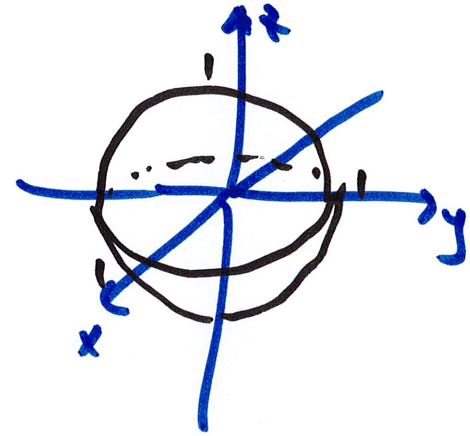
flux is: $\displaystyle \int_0^{2\pi}\int_0^{\pi}\int_{\varepsilon}^{1} \underbrace{\frac{2}{\rho}}_{\operatorname{div}\vec{F}} \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV}$

$$= \cdots = 4\pi(1-\varepsilon^2)$$

now take limit $\varepsilon \to 0$

$$\lim_{\varepsilon \to 0} 4\pi(1-\varepsilon)^2 = \boxed{4\pi}$$

example $\vec{F} = \dfrac{\langle x, y, z \rangle}{\sqrt{x^2+y^2+z^2}}$



S: sphere of radius 1

for the Divergence Theorem $\iint\limits_S \vec{F} \cdot \vec{n}\, dS = \iiint\limits_D \operatorname{div} \vec{F}\, dV$

to work, $\operatorname{div} \vec{F}$ MUST be defined in D.

$\operatorname{div} \vec{F} = \dfrac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \dfrac{2}{\sqrt{x^2+y^2+z^2}}$ clearly NOT defined at origin

to solve this, make up a tiny sphere with radius ε ← epsilon

then solve this like the double sphere example earlier,

then take $\lim\limits_{\varepsilon \to 0}$