14.1 Vector-Valued Functions

Scalar-valued functions: $f(t) = \underbrace{t^2+3}_{\text{scalar output}}$

scalar input

vector-~~fn~~ valued functions : $\vec{r}(t) = \underbrace{\langle \cos t, \sin t, 0 \rangle}_{\text{vector output}}$

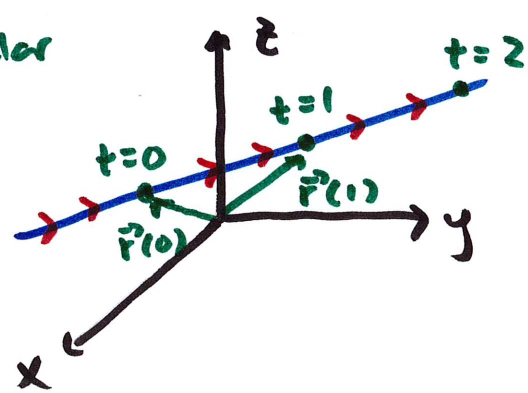
scalar input

we've seen vector-valued functions already

line through $P(1,2,3)$ and $Q(4,5,6)$

$$\vec{r}(t) = \langle 1,2,3 \rangle + t\langle 3,3,3 \rangle = \underbrace{\langle 1+3t, 2+3t, 3+3t \rangle}_{\text{vector}}$$

scalar

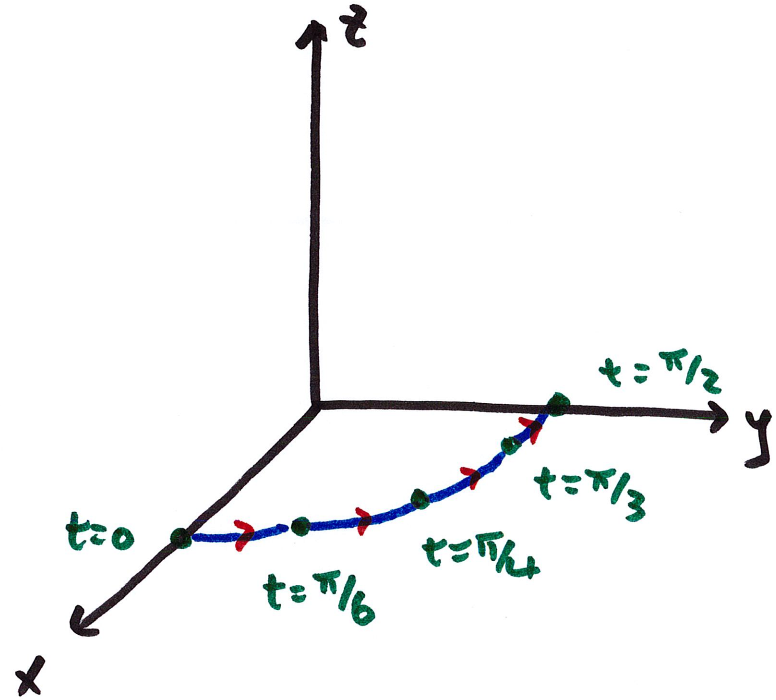


the direction of increasing t is called the <u>positive orientation</u>

example $\vec{r}(t) = \langle \cos t,\ \sin t,\ 0 \rangle \qquad 0 \le t \le \pi/2$

one way to visualize: plot points

t	x	y	z
0	1	0	0
$\pi/6$	$\sqrt{3}/2$	$1/2$	0
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	0
$\pi/3$	$1/2$	$\sqrt{3}/2$	0
$\pi/2$	0	1	0



part of circle moving counterclockwise viewed from above (z-axis coming at us) another way: analyze relationship between variables, connect to one of the surfaces we know

$$\vec{r}(t) = \langle \underbrace{\cos t}_{x(t)}, \underbrace{\sin t}_{y(t)}, \underbrace{0}_{z(t)} \rangle \quad 0 \le t \le \pi/2$$

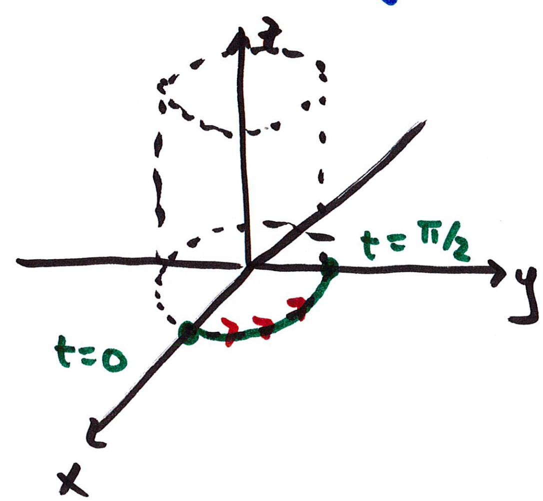
$x = \cos t$, $y = \sin t$, $z = 0$

we also know $\cos^2 t + \sin^2 t = 1$

so, we know in this case, $\underbrace{x^2 + y^2 = 1}_{\text{cylinder}}$, $z = 0$

this means this curve is on the surface of the cylinder with $z = 0$

don't ~~forn~~ forget: $0 \le t \le \pi/2$



example

$$\vec{F}(t) = \langle \underset{x}{0}, \ \underset{y}{\cos t}, \ \underset{z}{2\sin t} \rangle \qquad 0 \le t \le \pi/2$$

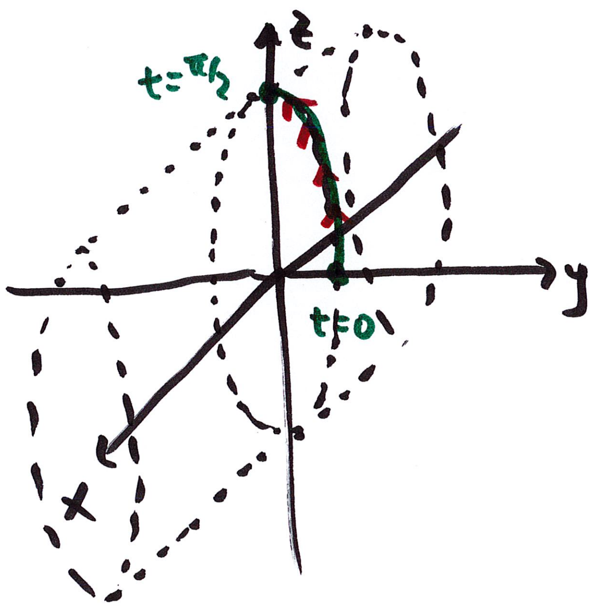
$$\left.\begin{array}{l} y = \cos t \\ z = 2\sin t \end{array}\right\} \text{ notice } \quad (2y)^2 + z^2 = 4\cos^2 t + 4\sin^2 t$$

$$= 4(\cos^2 t + \sin^2 t)$$

$$= 4$$

So, we recognize this curve is on the surface $4y^2 + z^2 = 4$

elliptical cylinder



$t = 0$: at $(0, 1, 0)$

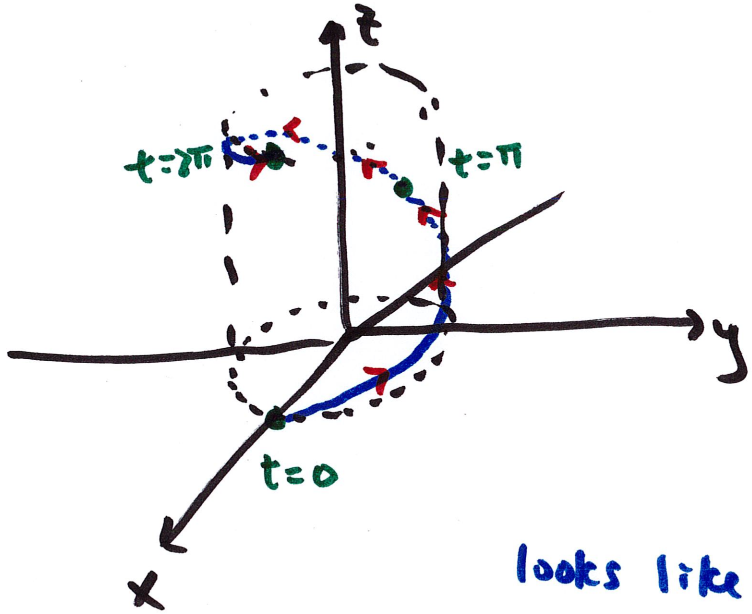
$t = \frac{\pi}{2}$: at $(0, 0, 2)$

example $\vec{r}(t) = \langle \cos t, \ \sin t, \ t \rangle$ $\quad 0 \le t \le 2\pi$

$\qquad\qquad\qquad\quad x \qquad \ \ y \qquad z$

$$\left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\} \ x^2 + y^2 = 1 \quad \text{circular cylinder}$$

$z = t$ → doesn't change the fact that $\vec{r}(t)$ is still on the cylinder, just means z is not fixed anymore



$t = 0$: at $(1, 0, 0)$

$t = \pi$: at $(-1, 0, \pi)$

$t = 2\pi$: at $(1, 0, 2\pi)$

looks like

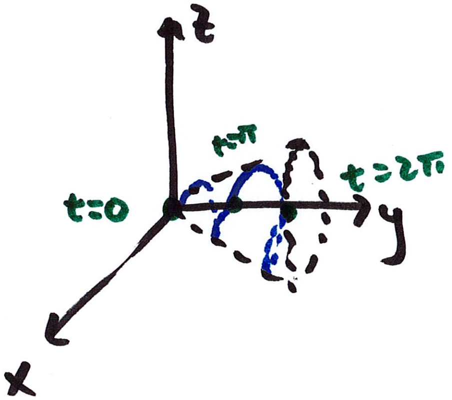
example Does the curve $\vec{r}(t) = \langle t\cos t,\ t,\ t\sin t \rangle \quad 0 \le t \le 2\pi$ intersect the plane $x - z = 0$?

what does $\vec{r}(t) = \langle \underset{x}{t\cos t},\ \underset{y}{t},\ \underset{z}{t\sin t} \rangle \quad 0 \le t \le 2\pi$ look like?

$$\left.\begin{array}{l} x = t\cos t \\ z = t\sin t \end{array}\right\} \quad x^2 + z^2 = t^2\cos^2 t + t^2 \sin^2 t = \underbrace{t^2}_{y}$$

$y = t$

so, $\vec{r}(t)$ is on the surface $x^2 + z^2 = y^2$ double cone



spiral ~~on~~ on surface of cone

if $\vec{r}(t) = \langle t\cos t,\ t,\ t\sin t \rangle$ intersects $x - z = 0$, then at some t, $\vec{r}(t)$ and $x - z = 0$ have the same x, y, z

$\vec{r}(t) = \langle \underset{x}{t\cos t},\ \underset{y}{t},\ \underset{z}{t\sin t} \rangle \qquad 0 \le t \le 2\pi$

$x - z = 0$

at ~~interse~~ intersection, we need $x - z = 0$

$t\cos t - t\sin t = 0 \qquad 0 \le t \le 2\pi$

$t(\cos t - \sin t) = 0$

$t = 0$ or $\cos t = \sin t$

$t = \pi/4,\ 5\pi/4$

so, the curve intersects $x - z = 0$ 3 times, at $t = 0,\ \pi/4,\ 5\pi/4$

points of intersection:

@ $t = 0$: $(0, 0, 0)$

@ $t = \pi/4$: $\left(\frac{\pi}{4\sqrt{2}}, \frac{\pi}{4}, \frac{\pi}{4\sqrt{2}}\right)$

@ $t = 5\pi/4$: $\left(-\frac{5\pi}{4\sqrt{2}}, \frac{5\pi}{4}, -\frac{5\pi}{4\sqrt{2}}\right)$

the domain of a vector-valued function is the interval ~~where~~ on which **ALL** components are defined.

example $\vec{r}(t) = \left\langle \sqrt{1-t^2},\ \sqrt{t},\ \dfrac{1}{\sqrt{5-t}} \right\rangle$

$\sqrt{1-t^2}$ is defined on $[-1, 1]$

$\sqrt{t}$ is defined on $[0, \infty)$

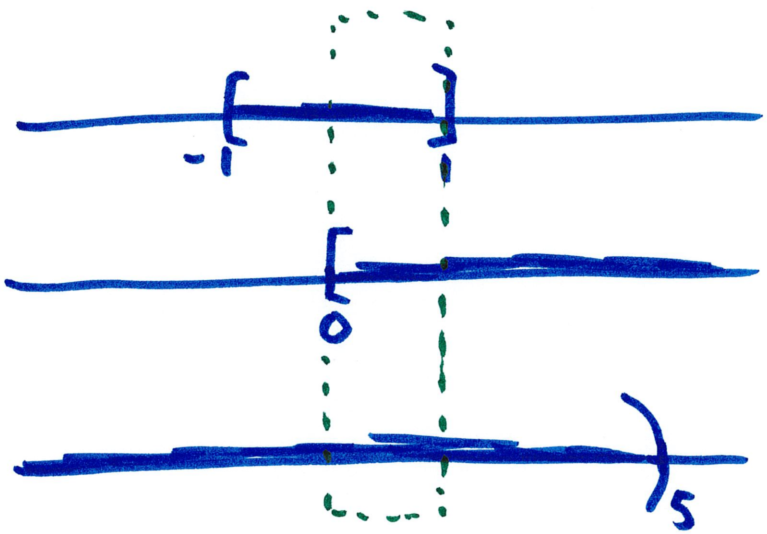
$\dfrac{1}{\sqrt{5-t}}$ is defined on $(-\infty, 5)$

intersection of these is where **ALL** 3 are defined

$\sqrt{1-t^2}$:

$\sqrt{t}$:

$\dfrac{1}{\sqrt{5-t}}$:



overlap on $[0, 1]$

so domain of $\vec{r}(t) = \left\langle \sqrt{1-t^2},\ \sqrt{t},\ \dfrac{1}{\sqrt{5-t}} \right\rangle$ is $[0, 1]$