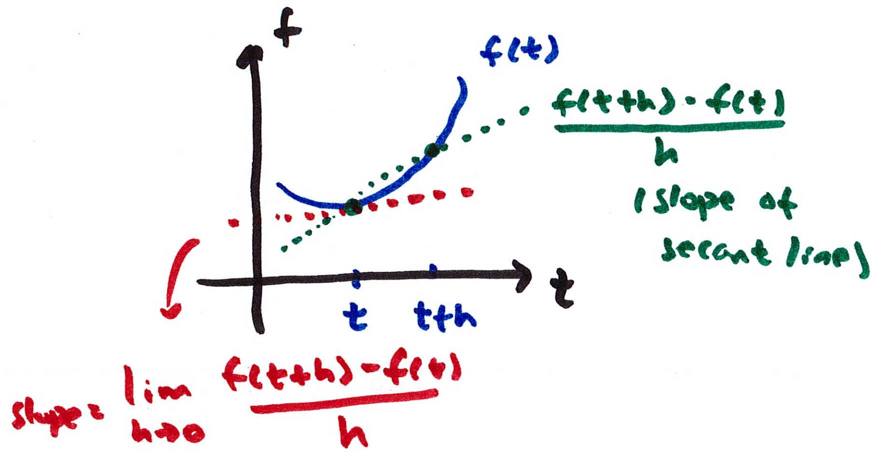
14.2 Calculus of Vector-Valued Functions

recall that if $f(t)$ is a scalar function

then $f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$



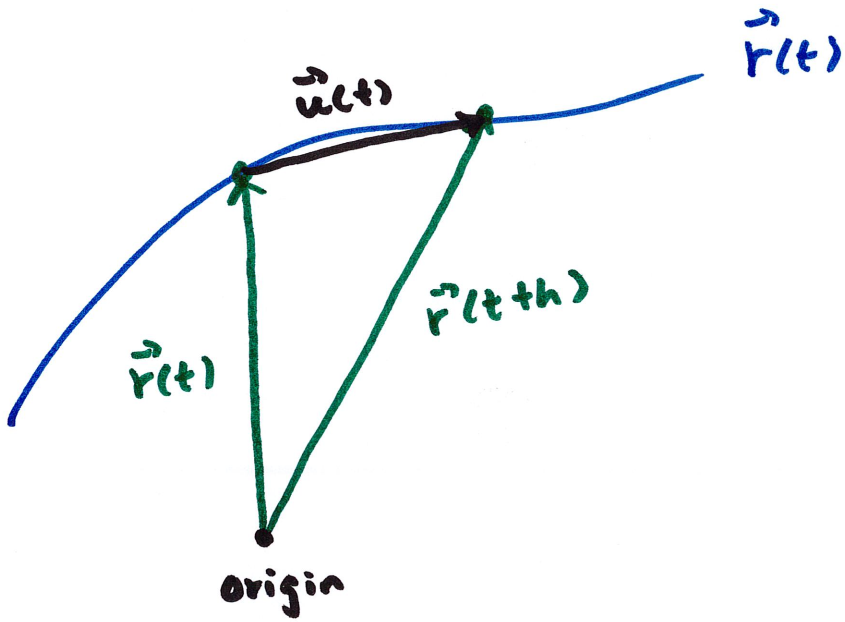
the idea w/ vector-valued function is the same

$\vec{r}(t) = \langle x(t), y(t) \rangle$

the derivative is defined the same way

$$\boxed{\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}}$$

the geometric interpretation is very similar to scalar case

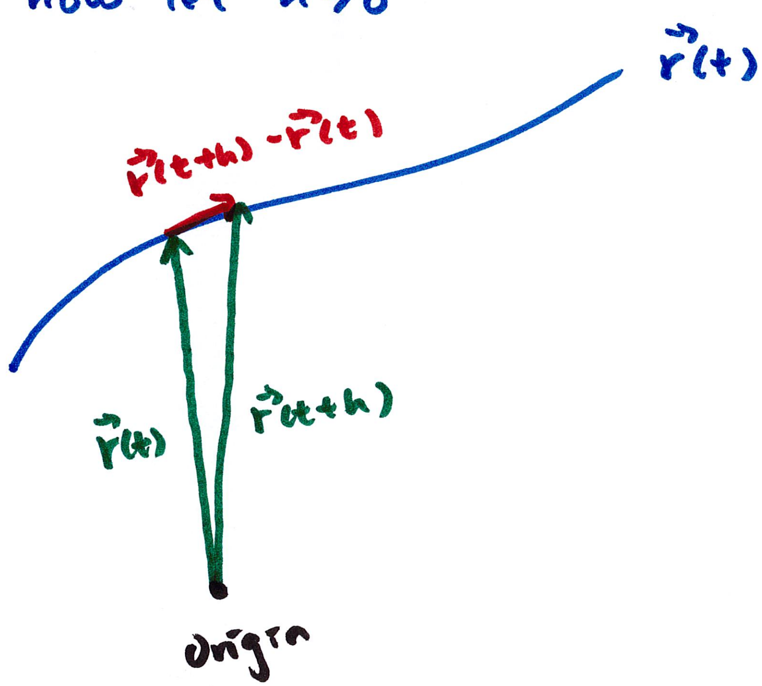


notice $\vec{r}(t)+\vec{u}(t)=\vec{r}(t+h)$

therefore $\vec{u}(t)=\vec{r}(t+h)-\vec{r}(t)$

(numerator of the difference quotient in $\vec{r}\,'$)

now let $h \to 0$



as expected, as $h \to 0$ $\vec{r}(t+h)-\vec{r}(t)$ eventually becomes tangent to the curve at t

so, again as expected,

$\vec{r}\,'(t)=\lim_{h \to 0} \dfrac{\vec{r}(t+h)-\vec{r}(t)}{h}$ gives us the tangent vector notice $\vec{r}\,'(t) = \lim_{h \to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$

let $\vec{r}(t) = \langle x(t), y(t) \rangle$

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\langle x(t+h), y(t+h) \rangle - \langle x(t), y(t) \rangle}{h}$$

$$= \lim_{h \to 0} \frac{\langle x(t+h) - x(t),\ y(t+h) - y(t) \rangle}{h} = \lim_{h \to 0} \left\langle \frac{x(t+h) - x(t)}{h},\ \frac{y(t+h) - y(t)}{h} \right\rangle$$

$$= \langle x'(t), y'(t) \rangle$$

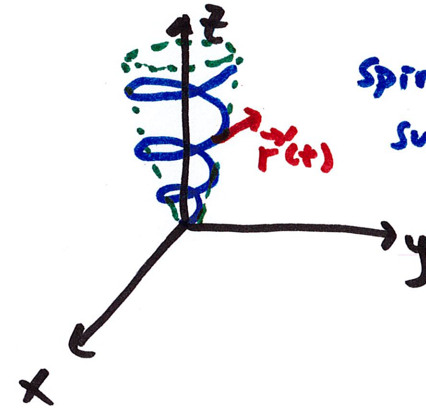
so, if $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$\underbrace{\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle}_{\text{tangent vector to the curve defined by } \vec{r}(t)}$$

example

$\vec{r}(t) = \langle t\cos t,\ t\sin t,\ t \rangle$

$\qquad\quad x \qquad\quad y \qquad z$



Spiral on surface of a cone

$x^2 = t^2\cos^2 t$

$y^2 = t^2 \sin^2 t$

$z^2 = t^2$

$x^2+y^2 = t^2(\cos^2 t + \sin^2 t) = t^2 = z^2 \longrightarrow$ surface $x^2+y^2=z^2$

$\vec{r}'(t) = \langle -t\sin t + \cos t,\ t\cos t + \sin t,\ 1 \rangle$

at, for example, at $t = \pi/2$

$\vec{r}'(\frac{\pi}{2}) = \langle -\frac{\pi}{2},\ 1,\ 1 \rangle$

$|\vec{r}'(t)| = \sqrt{(-t\sin t + \cos t)^2 + (t\cos t + \sin t)^2 + 1^2}$

$= \cdots$

$= \sqrt{t^2+2}$

the magnitude changes at t changes

in general, $\vec{r}\,'(t)$ is <u>not</u> a unit vector

but many things later in the course need to isolate the direction only

so, we define the unit tangent vector $\boxed{\vec{T} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}}$

for the example on the previous page,

$$\vec{T} = \left\langle \frac{-t\sin t + \cos t}{\sqrt{t^2+2}}, \frac{t\cos t + \sin t}{\sqrt{t^2+2}}, \frac{1}{\sqrt{t^2+2}} \right\rangle$$

many things, like the product rule, behave as expected.

<u>example</u> $\vec{r}(t) = \langle 1, t, t^2 \rangle$

$f(t) = e^t$

show that $\dfrac{d}{dt}\left[f(t)\vec{r}(t)\right] = f(t)\vec{r}\,'(t) + f'(t)\vec{r}(t)$

$\vec{r}(t) = \langle 1, t, t^2 \rangle$

$f(t) = e^t$

$$\frac{d}{dt}\left[f(t)\vec{r}(t)\right] = \frac{d}{dt}\langle e^t, te^t, t^2e^t \rangle$$

$$= \langle e^t, te^t + e^t, t^2e^t + 2te^t \rangle$$

$$= \langle 0, e^t, 2te^t \rangle + \langle e^t, te^t, t^2e^t \rangle$$

$$= \underset{\downarrow \atop f(t)}{e^t} \underbrace{\langle 0, 1, 2t \rangle}_{\vec{r}\,'(t)} + \underset{\downarrow \atop f'(t)}{e^t} \underbrace{\langle 1, t, t^2 \rangle}_{\vec{r}(t)}$$

Similarly, we can show that

$$\frac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \underbrace{\vec{u}\,'(t) \cdot \vec{v}(t)}_{\text{ok if written as } \vec{v} \cdot \vec{u}\,'} + \vec{u}(t) \cdot \vec{v}\,'(t)$$

note the order does not matter because $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

and

$$\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \underbrace{\vec{u}\,'(t) \times \vec{v}(t)}_{\text{not the same if written } \vec{v} \times \vec{u}} + \vec{u}(t) \times \vec{v}\,'(t)$$

order matters!

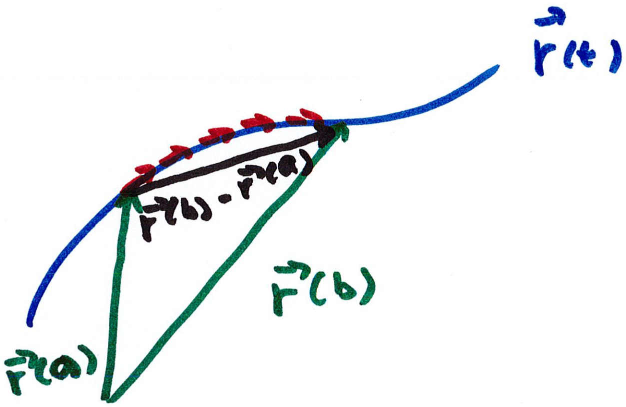
integrals behave mostly as expected

$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C}$$

such that $\vec{R}' = \vec{r}$

vector constant

$$\int_a^b \vec{r}(t)\,dt = \underbrace{\vec{R}(b) - R(a)}_{\text{displacement vector}}$$



$$\underbrace{\int_a^b \vec{r}'(t)\,dt}_{\text{add up all tangent vectors from a to b}} = \underbrace{\vec{r}(b) - \vec{r}(a)}_{\text{displacement}}$$

14.3 Motion in Space (part 1)

if $\vec{r}(t)$ is the position

then $\vec{r}\,'(t) = \vec{v}(t)$ (velocity) (vector)

and $|\vec{r}\,'(t)| = |\vec{v}(t)|$ is the speed (scalar)

$\vec{r}\,''(t) = \vec{v}\,'(t) = \vec{a}(t)$ is the acceleration (vector)

example If $\vec{a}(t) = \langle 1, t, t^2 \rangle$ is the acceleration vector $(t \geq 0)$

find the position $\vec{r}(t)$ such that

$\vec{v}(0) = \langle 1, 2, 3 \rangle$ (initial velocity)

and

$\vec{r}(0) = \langle 0, 0, 0 \rangle$ (initial position)

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle 1, t, t^2 \rangle\, dt$$

$$= \left\langle \int 1\,dt,\ \int t\,dt,\ \int t^2\,dt \right\rangle$$

$$\vec{v}(t) = \left\langle t + C_1,\ \tfrac{1}{2}t^2 + C_2,\ \tfrac{1}{3}t^3 + C_3 \right\rangle = \left\langle t,\ \tfrac{1}{2}t^2,\ \tfrac{1}{3}t^3 \right\rangle + \underbrace{\langle C_1, C_2, C_3 \rangle}_{\text{vector constant } \vec{C}}$$

use $\vec{v}(0)$ to find $\vec{C}$

$$\vec{v}(0) = \langle 0, 0, 0 \rangle + \vec{C} = \langle 1, 2, 3 \rangle \quad (\text{given } \vec{v}(0))$$

so, $\vec{C} = \langle 1, 2, 3 \rangle$

and $\boxed{\vec{v}(t) = \left\langle t+1,\ \tfrac{1}{2}t^2 + 2,\ \tfrac{1}{3}t^3 + 3 \right\rangle}$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \left\langle \tfrac{1}{2}t^2 + t,\ \tfrac{1}{6}t^3 + 2t,\ \tfrac{1}{12}t^4 + 3t \right\rangle + \overset{\langle d_1, d_2, d_3 \rangle}{\vec{D}}$$

$\vec{r}(0) = \langle 0, 0, 0 \rangle + \vec{D} = \langle 0, 0, 0 \rangle$ (given $\vec{r}(0)$) so, $\vec{D} = \langle 0, 0, 0 \rangle = \vec{0}$

so, $\boxed{\vec{r}(t) = \left\langle \tfrac{1}{2}t^2 + t,\ \tfrac{1}{6}t^3 + 2t,\ \tfrac{1}{12}t^4 + 3t \right\rangle}$