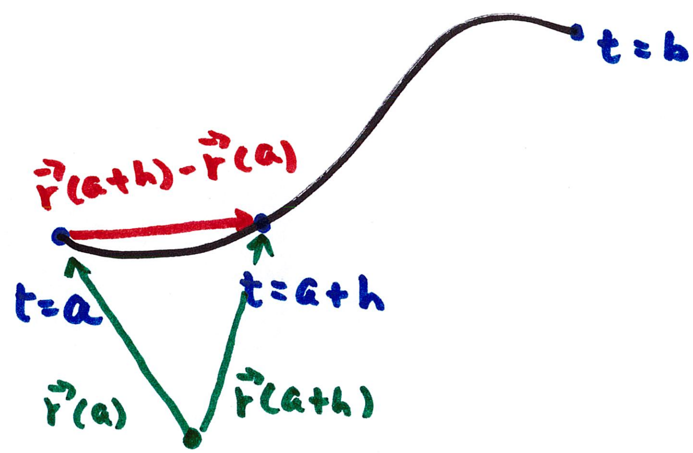
14.4 Length of Curves

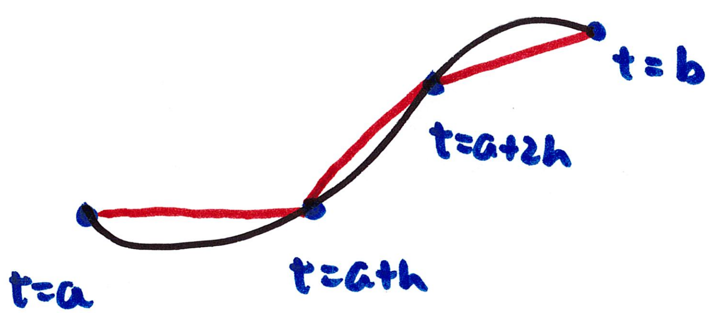
$\vec{r}(t)$, $a \le t \le b$

can approximate by using straight segments



if h is small, then ~~then~~ ~~then~~ the length of curve from $t=a$ to $t=a+h$ is approximately $|\vec{r}(a+h)-\vec{r}(a)|$

(the smaller h is, the better the approximation)



sum of approximations $\approx$ total length using segments

now we use calculus to shrink h and find the exact length from last time: $\vec{r}\,'(a) = \lim_{h \to 0} \dfrac{\vec{r}(a+h) - \vec{r}(a)}{h}$

so $\vec{r}\,'(a) \approx \dfrac{\vec{r}(a+h) - \vec{r}(a)}{h}$ when h is small

$$\vec{r}(a+h) - \vec{r}(a) \approx \vec{r}\,'(a)\,h$$

$$\underbrace{|\vec{r}(a+h) - \vec{r}(a)|}_{\text{the line segment approx. at } t=a} \approx |\vec{r}\,'(a)|\,h \qquad (\text{assume } h>0)$$

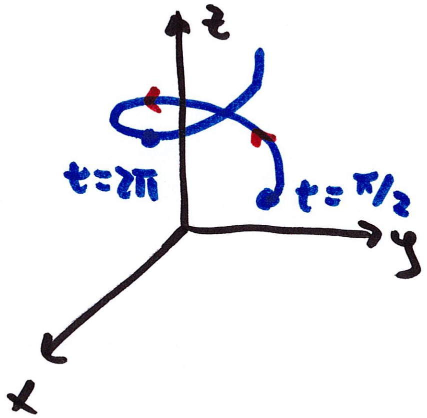
now, shrink $h \to 0$ then $h \to dt$

and summing up all the little segments from $t=a$ to $t=b$, we integrate

the exact length of $\vec{r}(t)$, $a \le t \le b$ is therefore

$$\boxed{L = \int_a^b |\vec{r}\,'(t)|\,dt}$$

<u>example</u> $\vec{r}(t) = \langle \cos t, \sin t, t \rangle \qquad \frac{\pi}{2} \le t \le 2\pi$

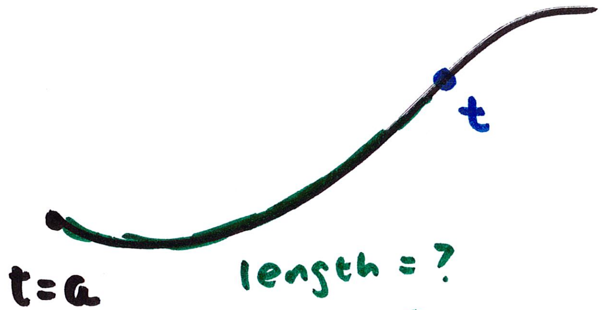


$$L = \int_a^b |\vec{r}\,'(t)|\,dt \qquad \vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$= \int_{\pi/2}^{2\pi} \sqrt{2}\,dt = \sqrt{2}\left(2\pi - \frac{\pi}{2}\right) = \frac{3\pi}{2}\cdot\sqrt{2} = \boxed{\frac{3\pi}{\sqrt{2}}}$$ exact length

we can tweak it a bit to find the <u>length as a function of t</u>



don't specify b, leave as t

$$L(t) = \int_a^t |\vec{r}\,'(u)|\,du$$

u: "dummy variable" use it instead of t because t is the parameter of the length function

normally we use "s" for the length function

$$\boxed{s(t) = \int_a^t |\vec{r}\,'(u)|\,du}$$

try it on the helix : $\vec{r}(t) = \langle \cos t, \sin t, 1 \rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 0 \rangle \quad |\vec{r}\,'(t)| = \sqrt{2} = |\vec{r}\,'(u)|$

$$s(t) = \int_{\pi/2}^{t} \sqrt{2}\,du = \boxed{\sqrt{2}\left(t - \frac{\pi}{2}\right)}$$

length of $\vec{r}(t)$ from $t = \pi/2$ to some t

notice $s(2\pi) = \sqrt{2}\left(2\pi - \frac{\pi}{2}\right)$ is, as expected, exact equal to $\int_{\pi/2}^{2\pi} |\vec{r}\,'(t)|\,dt$

$s(t) = \int_a^t |\vec{r}\,'(u)|\,dt$ tells us s and t are related

length   often time

$\vec{r}(t) = \langle \cos t, \sin t, 1 \rangle$ tells us where we are at a <u>given time</u>

but if we can somehow change the parameter to s

then $\vec{r}(s)$ tells us where we are <u>after having traveled a length of s on the curve</u>

revisit the helix example

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle \qquad t \geq \frac{\pi}{2}$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle \qquad |\vec{r}\,'(t)| = \sqrt{2}$ and $\underbrace{s(t) = \sqrt{2}\,(t - \tfrac{\pi}{2})}$

$\frac{s}{\sqrt{2}} = t - \frac{\pi}{2}$

or $t = \frac{s}{\sqrt{2}} + \frac{\pi}{2}$

$$\boxed{\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}} + \frac{\pi}{2}\right), \sin\left(\frac{s}{\sqrt{2}} + \frac{\pi}{2}\right), \frac{s}{\sqrt{2}} + \frac{\pi}{2} \right\rangle}$$

from $\vec{r}(t)$: $\vec{r}(2\pi)$ is where we are at $t = 2\pi$

from $\vec{r}(s)$: $\vec{r}\left(\frac{3\pi}{\sqrt{2}}\right)$ is where we are having traveled a length of $\frac{3\pi}{\sqrt{2}}$ on curve

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle \quad \frac{\pi}{2} \le t \le 2\pi$

$\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}} + \frac{\pi}{2}\right), \sin\left(\frac{s}{\sqrt{2}} + \frac{\pi}{2}\right), \frac{s}{\sqrt{2}} + \frac{\pi}{2} \right\rangle \quad 0 \le s \le \frac{3\pi}{\sqrt{2}}$

from $s(t) = \sqrt{2}\left(t - \frac{\pi}{2}\right)$

also, notice that from $s(t) = \int_a^t |\vec{r}\,'(u)|\,du$

if $\frac{ds}{dt} = 1$ $\rightarrow$ length and time change at the same rate

then $\frac{ds}{dt} = \underbrace{\frac{d}{dt}\int_a^t |\vec{r}\,'(u)|\,du = |\vec{r}\,'(t)|}_{\text{Fundamental Theorem of Calculus}} = 1$

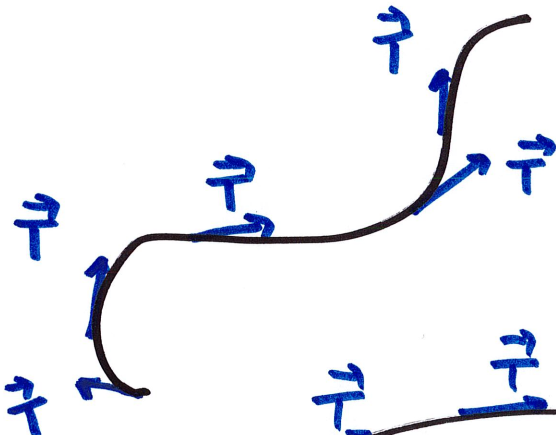
if this is the case, then $\vec{r}(t)$ and $\vec{r}(s)$ will look the same

(so, in $\vec{r}(t)$, that t could also be length)

14.5 Curvature

the unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

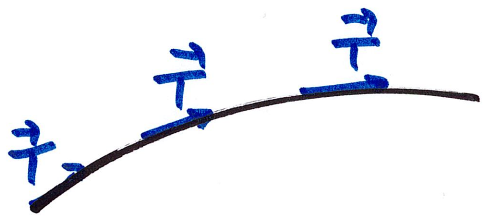
since $|\vec{T}| = 1$ by definition, and change in $\vec{T}(t)$ must be from the change in direction alone.



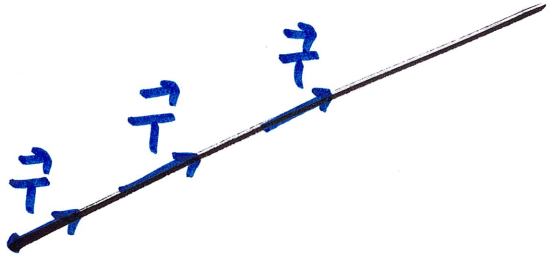
notice $\vec{T}$ changes direction often

so, we expect $\left|\dfrac{d\vec{T}}{ds}\right|$ to be big

length



smaller $\left|\dfrac{d\vec{T}}{ds}\right|$



$\vec{T}$ never changes direction

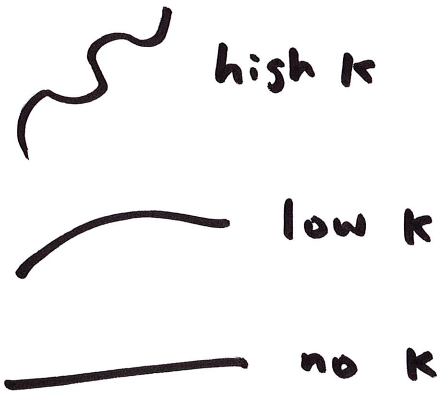
so $\left|\dfrac{d\vec{T}}{ds}\right| = 0$

$\left|\dfrac{d\vec{T}}{ds}\right|$ measures how sharply $\vec{T}$ turns as we travel on $\vec{r}$

we call $\left|\frac{d\vec{T}}{ds}\right|$ the <u>curvature</u> and usually give it the symbol κ

Greek letter "kappa"

$$\kappa = \left|\frac{d\vec{T}}{ds}\right|$$



in practice $\left|\frac{d\vec{T}}{ds}\right|$ is not always easy to use, since $\vec{T}$ is normally function of t

so, from $\frac{ds}{dt} = |\vec{r}'(t)|$ we can rewrite $\left|\frac{d\vec{T}}{ds}\right|$ as

$$\kappa = \frac{|d\vec{T}|}{|ds|} = \frac{|d\vec{T}/dt|}{|\vec{r}'(t)\,dt|} = \text{Y drT} \quad \boxed{\frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \kappa}$$

other formulas for κ :

$$\kappa = \frac{|\vec{r}\,'' \times \vec{r}\,'|}{|\vec{r}\,'|^3}$$

and if $\vec{r}\,' = \vec{v}$ and $\vec{r}\,'' = \vec{a}$

then $$\kappa = \frac{|\vec{a} \times \vec{v}|}{|\vec{v}|^3}$$