15.8 Lagrange Multipliers

constrained optimization problems : find max/min of something subject to a certain condition

for example, find max/min of $f(x,y)=x^2+y^2$ subject to the condition that $xy=1$

$f(x,y)=x^2+y^2$ is called the <u>objective</u>

$g(x,y)=xy-1=0$ is called the <u>constraint</u>

Since the constraint is simple we can solve this by substitution

$f(x,y)=x^2+y^2$ $\qquad xy=1 \rightarrow y=\frac{1}{x}$

$f(x,y)=x^2+\left(\frac{1}{x}\right)^2=x^2+\frac{1}{x^2}=f(x)$

$f'(x)=2x-\frac{2}{x^3}=0 \rightarrow x^4=1 \rightarrow x=-1,\ x=1$

$y=\frac{1}{x} \rightarrow y=-1$ when $x=-1$, $y=1$, $x=1$

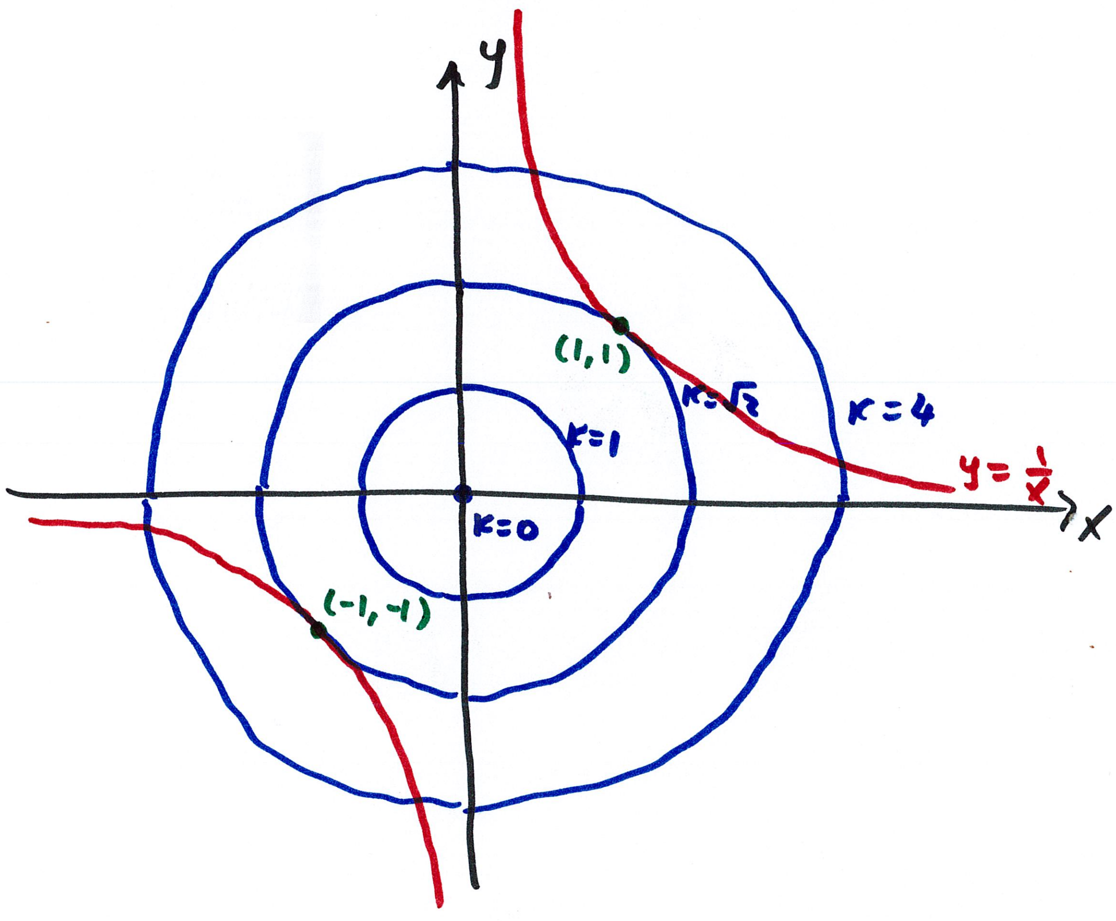
critical pts: $(-1,-1)$, $(1,1)$

$f(x,y) = x^2 + y^2 \qquad xy = 1$

f does not have a maximum since we can make x or y arbitrarily large
($y = \frac{1}{x}$, or $x = \frac{1}{y}$)

so critical pts (1, 1), (−1, −1) must be where minima are

the geometric interpretation is much more important



$f(x,y) = x^2 + y^2$

level curves: $x^2 + y^2 = k$

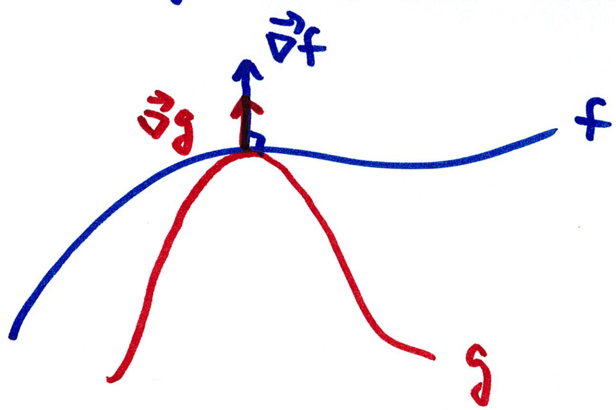
constraint: $xy = 1$

$y = \frac{1}{x}$ (hyperbolas)

at (1,1) any movement on the red curve → bigger k

at locations of max/min, the constraint curve and the objective level curve are <u>tangent</u> to each other at the point where the objective (f) is tangent to the constraint (g)

their gradients must also be tangent



tangent gradients $\Rightarrow$ $\vec{\nabla} f = \lambda \vec{\nabla} g$

Greek lowercase "lambda" (Lagrange multiplier)

Method of Lagrange Multipliers: to find max/min of $f(x,y)$ subject to constraint $g(x,y) = 0$ find where $\vec{\nabla} f = \lambda \vec{\nabla} g$

Example

$f(x,y) = 4 - x^2 - y^2$

subject to $4x^2 + y^2 = 4$

rewrite: $g(x,y) = 4x^2 + y^2 - 4 = 0$

Solve $\vec{\nabla} f = \lambda \vec{\nabla} g$

$\vec{\nabla} f = \langle f_x, f_y \rangle = \langle -2x, -2y \rangle$

$\vec{\nabla} g = \langle g_x, g_y \rangle = \langle 8x, 2y \rangle$

$\vec{\nabla} f = \lambda \vec{\nabla} g \rightarrow \langle -2x, -2y \rangle = \lambda \langle 8x, 2y \rangle$

$-2x = \lambda \cdot 8x$ — ①

$-2y = \lambda \cdot 2y$ — ②

} solve for x, y and optionally λ with the help of $g(x,y) = 0$

①: $\lambda \cdot 8x + 2x = 0$

$2x(4\lambda + 1) = 0 \rightarrow x = 0$ or $\lambda = -1/4$

↳ using $g(x,y) = 4x^2 + y^2 - 4 = 0$ we get $y = \pm 2$

points of interest: $(0, 2), (0, -2)$

②: $\lambda \cdot 2y + 2y = 0$

$2y(\lambda+1) = 0 \rightarrow y = 0$ or $\lambda = -1$

↳ from $g(x,y) = 4x^2+y^2-4 = 0$

we get $x = \pm 1$

points of interest: $(1,0)$, $(-1,0)$

now compare $f(x,y) = 4-x^2-y^2$ at the points of interest

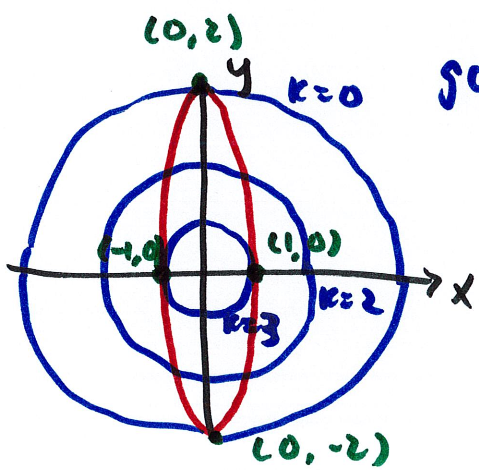
$f(0,2) = 0$
$f(0,-2) = 0$ } two minima at $(0,\pm 2)$, $f = 0$

$f(1,0) = 3$
$f(-1,0) = 3$ } two maxima at $(\pm 1, 0)$, $f = 3$

geometric view: $f(x,y) = 4-x^2-y^2 \rightarrow$ level curves are circles

$k = 4-x^2-y^2 \rightarrow x^2+y^2 = (4-k)$

$g(x,y) = 4x^2+y^2-4 = 0 \rightarrow$ ellipse



<u>example</u> $f(x,y,z) = xyz$

Subject to $x^2+y^2+z^2=3 \longrightarrow g(x,y,z) = x^2+y^2+z^2-3=0$

geometric interpretation of this example: find places on the sphere $x^2+y^2+z^2=3$ where the product of the coordinates (x,y,z) is max/min

Solve $\vec{\nabla} f = \lambda \vec{\nabla} g$

$\vec{\nabla} f = \langle yz, xz, xy \rangle$

$\vec{\nabla} g = \langle 2x, 2y, 2z \rangle$

$\vec{\nabla} f = \lambda \vec{\nabla} g \rightarrow \langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$

$yz = \lambda \cdot 2x$ — ① $\rightarrow \lambda = \frac{yz}{2x}$

$xz = \lambda \cdot 2y$ — ② $\rightarrow \lambda = \frac{xz}{2y}$

$xy = \lambda \cdot 2z$ — ③ $\rightarrow \lambda = \frac{xy}{2z}$

$\left.\right\} \frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z}$

$g = x^2+y^2+z^2-3=0$ — ④

$$\frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z}$$

$y^2 z = x^2 z \;\Rightarrow\; x^2 = y^2$

$xz^2 = xy^2 \;\Rightarrow\; y^2 = z^2$

$$x^2 = y^2 = z^2$$

now sub into $g(x, y, z) = x^2 + y^2 + z^2 - 3 = 0$

$3x^2 - 3 = 0 \rightarrow x = \pm 1$

so $y = \pm 1$

$z = \pm 1$

points of interest:

$(1, 1, 1)$

$(-1, -1, -1)$

$(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$

$(1, 1, -1)$, $(1, -1, 1)$, $(-1, 1, 1)$

compare $f(x, y, z) = xyz$ at these locations

max: 1 at $(1, 1, 1)$, $(-1, -1, 1)$, $(-1, 1, -1)$, $(+1, -1, -1)$

min: -1 at the other places

when we solved for λ earlier we implicitly required $x \neq 0$, $y \neq 0$, $z \neq 0$

but these are possible locations on the sphere

however, $f = xyz$, if one is zero $f = 0$ and it's neither max nor min