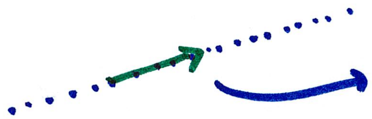
13.5 Lines and Planes

line: collection of points that lie along a certain direction

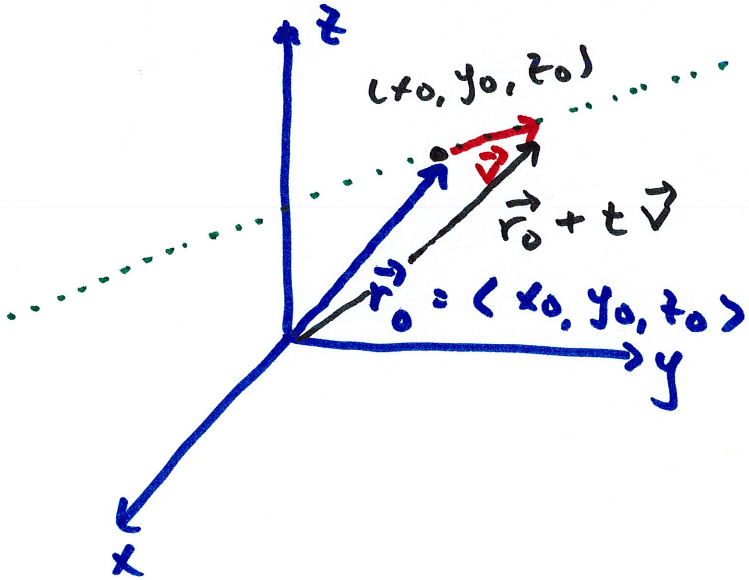


→ the direction vector (like slope in $\mathbb{R}^2$)

to find the direction vector, find the position vector of one point from any other

if we know one point on the line: (x_0, y_0, z_0)

how to write the equation of this line → finding the position vector of any point on this line



$\vec{r}_0$: vector from origin to (x_0, y_0, z_0)

$\vec{v}$: direction vector

any other point is reached by

$$\boxed{\vec{r}(t) = \vec{r}_0 + t\,\vec{v}} \qquad -\infty < t < \infty$$

$\vec{r}(t) = \vec{r}_0 + t\,\vec{v}$ is called the <u>vector form</u> of eq. of a line

if $\vec{v} = \langle a, b, c \rangle$

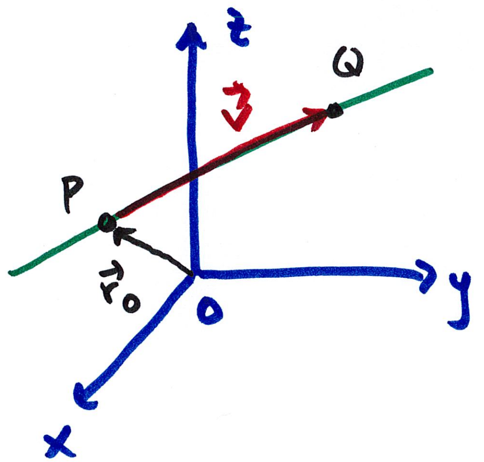
$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

then $\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$

$= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$

$$\left.\begin{aligned} x(t) &= x_0 + at \\ y(t) &= y_0 + bt \\ z(t) &= z_0 + ct \end{aligned}\right\} \quad \underline{\text{parametric form}}$$

<u>example</u> Line through $P(0,1,2)$ $Q(-3,4,7)$



direction vector : some multiple of $\vec{PQ}$ or $\vec{QP}$

let's do $\vec{PQ}$: $\vec{v} = \vec{PQ} = \langle -3, 3, 5 \rangle$

$\vec{r}_0$: from orgin to any known point

let's do $\vec{r}_0 = \vec{OP} = \langle 0, 1, 2 \rangle$

then $\vec{r}(t) = \vec{r}_0 + t\,\vec{v}$

$= \langle 0, 1, 2 \rangle + t \langle -3, 3, 5 \rangle$ vector form

$= \langle -3t, 1+3t, 2+5t \rangle$

$x = -3t$

$y = 1+3t$ parametric form

$z = 2+5t$

$-\infty < t < \infty$ infinitely long line if we just want the segment from P to Q, restrict t

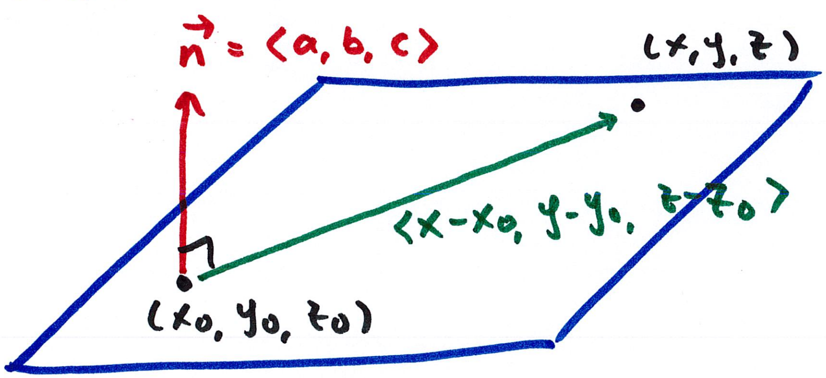
notice if $t=0$, $\vec{r}(0) = \langle 0, 1, 2 \rangle$ tip is at P

notice if $t=1$, $\vec{r}(1) = \langle 0, 1, 2 \rangle + \langle -3, 3, 5 \rangle$

$= \langle -3, 4, 7 \rangle$ tip is at Q

so, $0 \le t \le 1$ we get the segment $\vec{PQ}$

Plane :



(x_0, y_0, z_0) : known point

(x, y, z) : some ~~p~~ other point

vector from (x_0, y_0, z_0) to (x, y, z) is orthogonal to a vector perpendicular to plane ("normal vector")

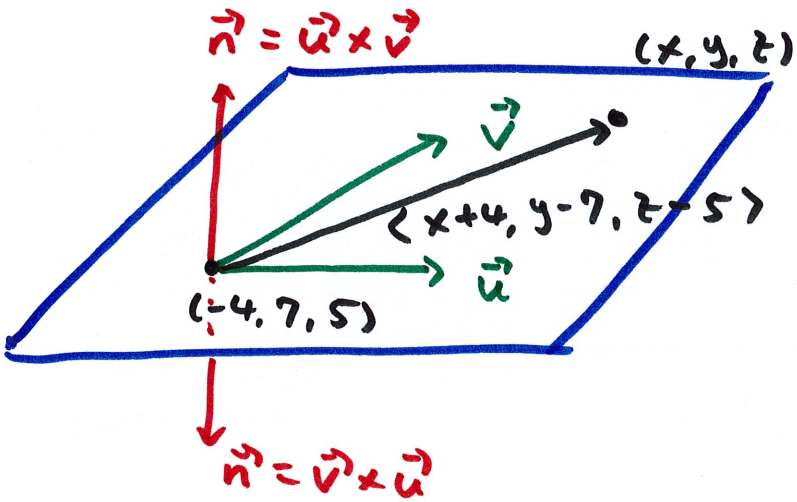
we know $\vec{n} \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$

$\langle a, b, c \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

eq. of plane through (x_0, y_0, z_0) with normal vector $\vec{n} = \langle a, b, c \rangle$

example Plane containing the vectors $\vec{u} = \langle 0, 1, 2 \rangle$, $\vec{v} = \langle -1, -3, 0 \rangle$ and point $(-4, 7, 5)$



$\vec{n}$ is perpendicular to BOTH $\vec{u}, \vec{v}$

so, $\vec{n} = \vec{u} \times \vec{v}$ or $\vec{v} \times \vec{u}$

let's do $\vec{n} = \vec{u} \times \vec{v}$

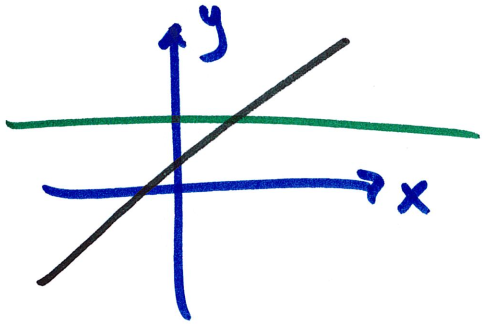
$= \cdots = \langle 6, -2, 1 \rangle$

$\vec{n} \cdot \langle x+4, y-7, z-5 \rangle = 0$

$\langle 6, -2, 1 \rangle \cdot \langle x+4, y-7, z-5 \rangle = 0$

$$\boxed{6(x+4) - 2(y-7) + (z-5) = 0}$$

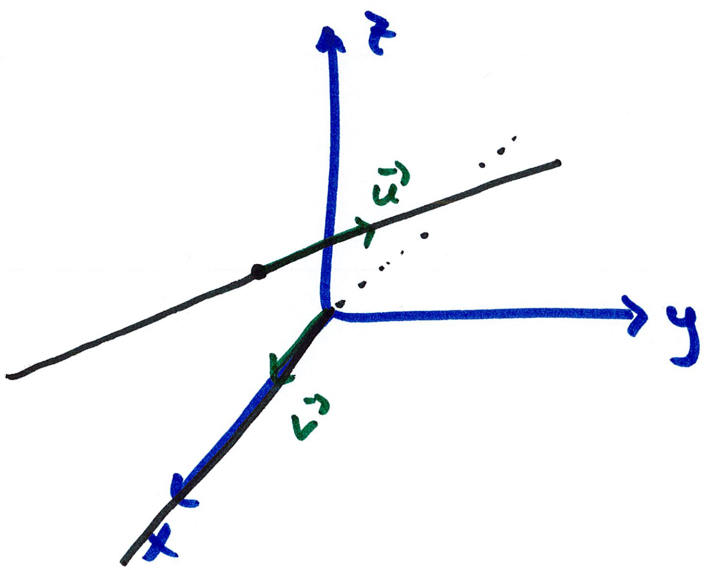
~~if~~ in $\mathbb{R}^2$, if two lines do not have same slope, they must intersect



that is NOT true in $\mathbb{R}^3$ (or higher)

e.g. $\vec{u} = \langle 1, 0, 1 \rangle + t \langle 1, 2, 3 \rangle$

$\vec{v} = t \langle 1, 0, 0 \rangle$

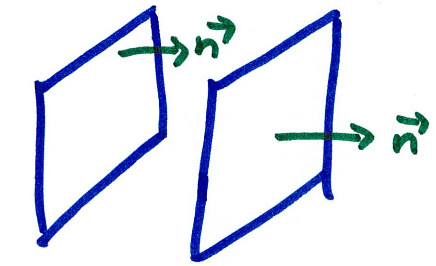


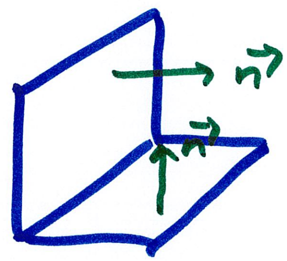
these lines never intersect (gap in z direction)

we say these lines are <u>skew</u>

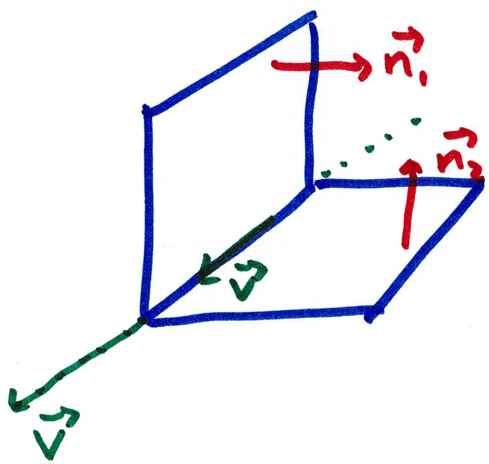
Similarly, two planes are parallel if their normal vectors are parallel

two planes are orthogonal if their normal vectors are ~~orgotho~~ orthogonal





interection of two planes is a line



$\vec{v}$, the direction vector of line of intersection is $\perp$ to both of the normal vectors

$\vec{v} = \vec{n}_1 \times \vec{n}_2$ or $\vec{n}_2 \times \vec{n}_1$

line of intersection contains pts that are on both planes if given plane eq, normal vectors can be found by looking at coefficients of x, y, z

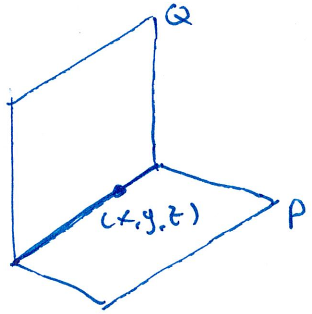
$$6(x+4) - 2(y-7) + 1(z-5) = 0$$

$\vec{n} = \langle 6, -2, 1 \rangle$ or $\langle -6, 2, -1 \rangle$ or any multiple of either extra

How to find a point on the line of intersection of two planes?

plane Q: $x+3y-2z=1$

plane R: $x+y+z=0$



we need x,y,z such that $x+3y-2z=1$ <u>AND</u> $x+y+z=0$ are BOTH true

we have 3 variables (x,y,z) and 2 constraints, so one variable is arbitrary (we can choose whatever we want)

as an example, I choose $z=0$

to find x,y, I solve $x+3y-2\cancel{z}^{\,0}=1$ and $x+y+\cancel{z}^{\,0}=0$ simultaneously

Supplemental example

example Two objects travel on the lines

$$\vec{r}_1(t) = \langle 2t+3,\ 4t+2,\ 3t+5 \rangle \quad -\infty < t < \infty$$

$$\vec{r}_2(s) = \langle s+2,\ 3s-1,\ -5s+10 \rangle \quad -\infty < s < \infty$$

will the objects' paths intersect?

will the objects collide with each other?

intersect : can we find t, s such that $\vec{r}_1(t) = \vec{r}_2(s)$?

collide : can we find t, s such that $\vec{r}_1(t) = \vec{r}_2(s)$ AND $t = s$?

if $\vec{r}_1(t) = \vec{r}_2(s)$ then

x: $2t+3 = s+2$ — ①

y: $4t+2 = 3s-1$ — ②

z: $3t+5 = -5s+10$ — ③

from ①, $s = 2t+1$

sub into ② $4t+2 = 3(2t+1)-1 \rightarrow \boxed{t = 0 \quad \text{so,} \quad s = 1}$ check if these work in ③

③ : $3(0)+5 = -5(1)+10$?

yes, so at $t=0$, $s=1$, $\vec{r}_1(t) = \vec{r}_2(s)$ they intersect