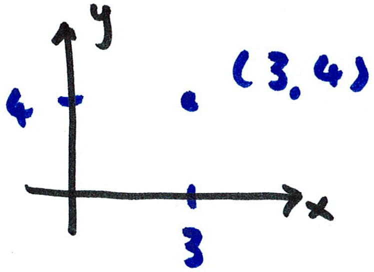
16.3 Double Integrals in Polar Coordinates

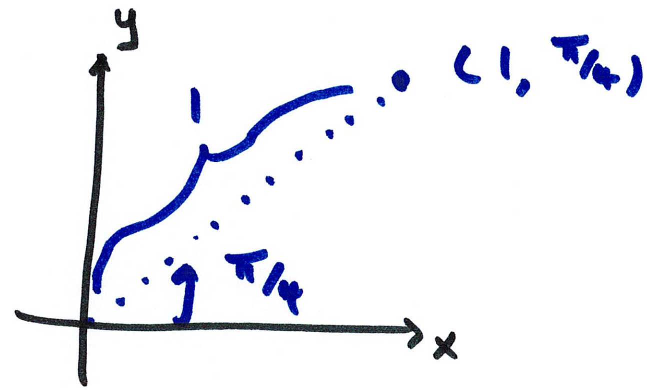
in rectangular coordinates, a point is located as (x, y)



Polar: (r, θ)

r → displacement from origin

θ → angle of line thru origin and the point

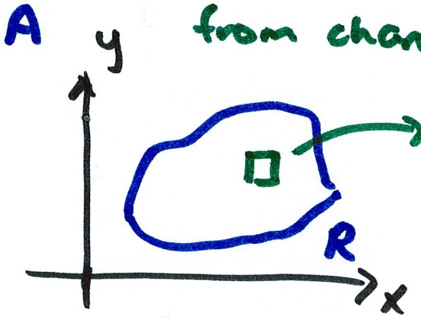


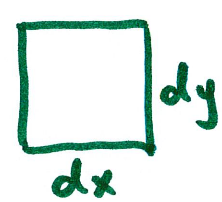
conversion: $x^2+y^2=r^2$

$x = r\cos\theta$

$y = r\sin\theta$

In Cartesian, $\iint_R f(x,y)\,dA$ → area of small patch of region R from change in x and change in y

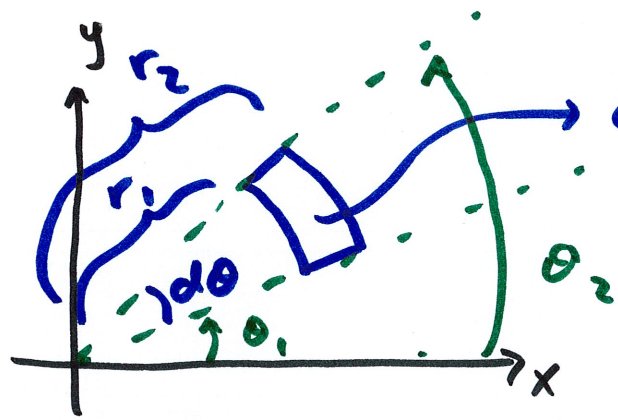




$dA = dx\,dy$ or $dy\,dx$

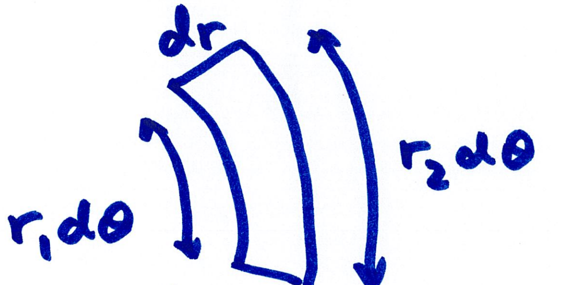
in polar, $\iint_R f(r,\theta)\, dA$

↪ area of a small patch of R from change in r, θ



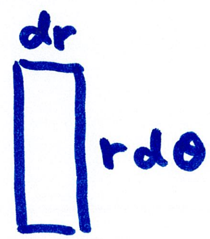
area = dA

note this is not a rectangle



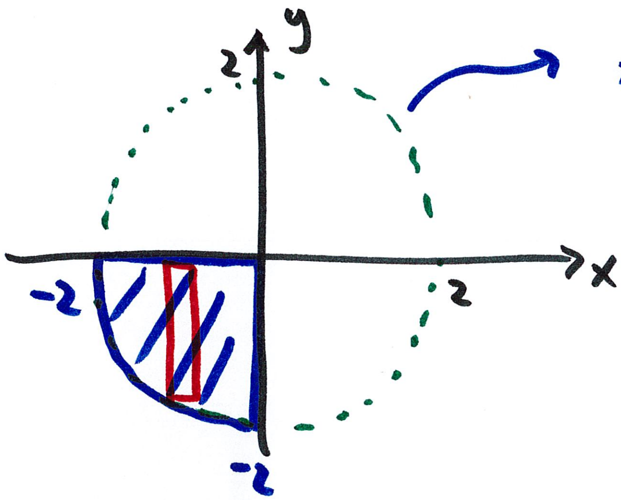
area of this ?

when we integrate we sum up infinitely-many of these so $d\theta$ is small and dr is small $\rightarrow r_1 \approx r_2 = r$

$\approx$ 

area is $\boxed{dA = r\,d\theta\,dr = r\,dr\,d\theta}$

example $\displaystyle\iint_R \cos(x^2+y^2)\,dA$ R : circle centered at (0,0) radius 2 in 3rd quadrant



$x^2+y^2=4$

$y=\sqrt{4-x^2}$ upper half

$y=-\sqrt{4-x^2}$ lower half

Set up as Type I region: $R=\{(x,y) : -2 \le x \le 0,\ -\sqrt{4-x^2} \le y \le 0\}$

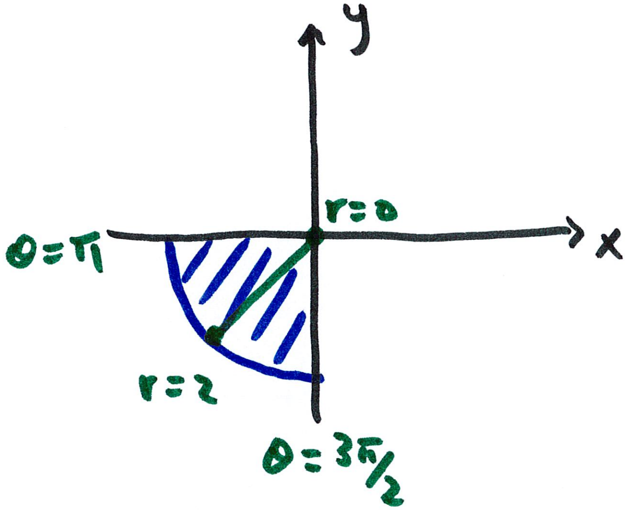
integral in Cartesian:

$$\int_{-2}^{0}\int_{-\sqrt{4-x^2}}^{0} \cos(x^2+y^2)\,dy\,dx$$

this is hard to ~~evalute~~ evaluate

try this in polar

in polar,



$$R = \left\{ (r,\theta) : \ 0 \le r \le 2, \quad \pi \le \theta \le \frac{3\pi}{2} \right\}$$

$$\iint_R \cos(\underbrace{x^2+y^2}_{\text{in polar is } r^2})\, \overbrace{dA}^{\text{in polar is } = r\,dr\,d\theta}$$

$$= \int_{\pi}^{\frac{3\pi}{2}} \int_0^2 \cos(r^2)\, r\,dr\,d\theta = \int_{\pi}^{\frac{3\pi}{2}} \int_0^2 r\cos(r^2)\,dr\,d\theta$$

θ $\quad$ r

$u = r^2 \qquad du = 2r\,dr$

$$= \cdots = \boxed{\frac{\pi}{4}(\sin 4)}$$

example $\displaystyle\int_0^1 \int_{\sqrt{x-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx$

in Cartesian, this is terrible

let's examine the region R

$0 \le x \le 1$ $\qquad \underbrace{\sqrt{x-x^2}} \le y \le \underbrace{\sqrt{1-x^2}}$

$y = \sqrt{x-x^2}$

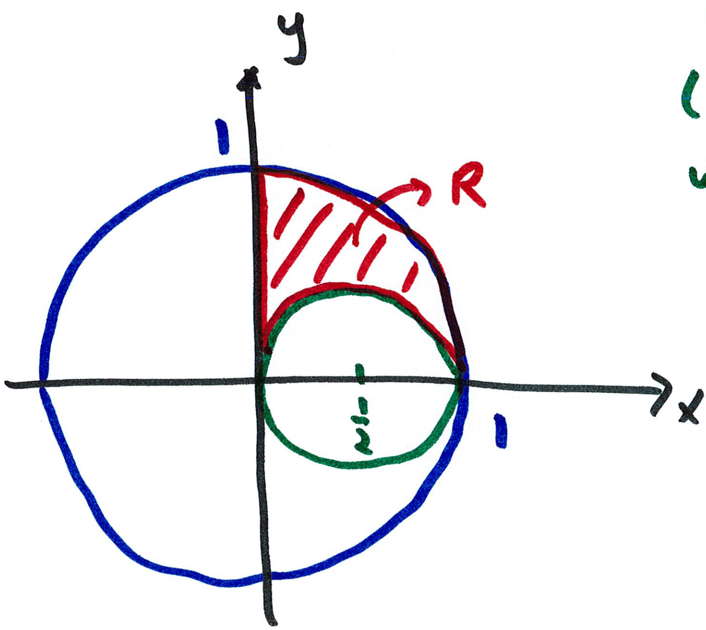
upper half circle centered at origin radius 1

$y^2 = x - x^2$

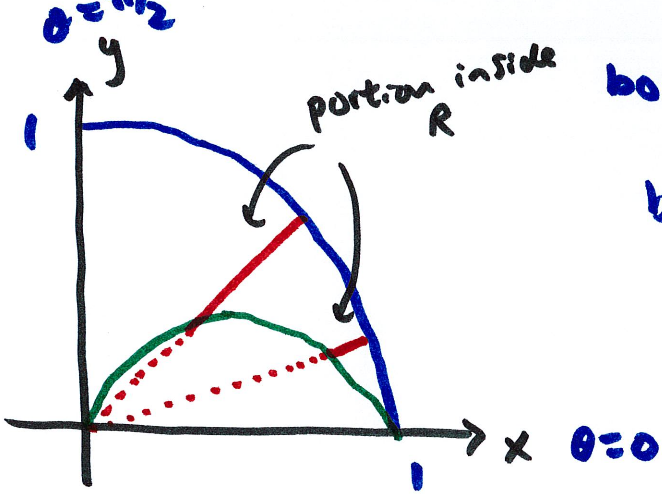
$x^2 - x + y^2 = 0$

$(x-\frac{1}{2})^2 + y^2 = \frac{1}{4}$

upper half of circle radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0)$



rewrite R in polar



bounds for θ is easy: $0 \le \theta \le \pi/2$

bounds for r: draw a line from origin to outer edge of R and see the portion of line inside R

$$\underbrace{\text{small circle}}_{} \le r \le \underbrace{\text{big circle}}_{}$$

equation in polar

$y^2 = x - x^2$

$\underbrace{x^2 + y^2}_{r^2} = \underbrace{x}_{r\cos\theta}$

$r^2 = r\cos\theta$

$\boxed{r = \cos\theta}$

equation in polar: $\boxed{r = 1}$

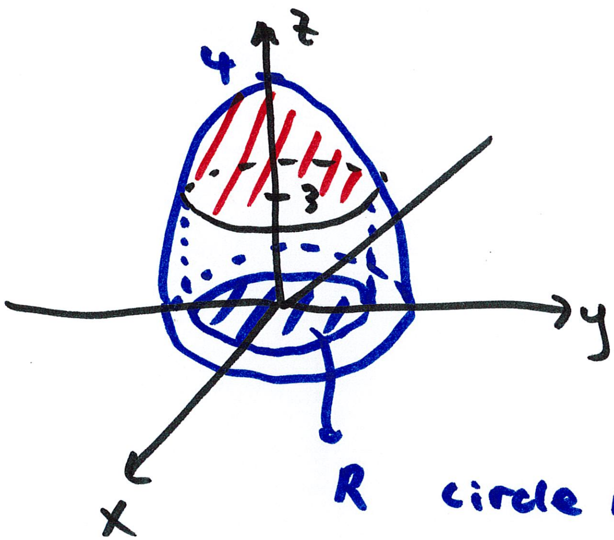
integrate this first because not bounded by constants

$$\boxed{R = \{(r, \theta), \quad 0 \le \theta \le \pi/2, \quad \cos\theta \le r \le 1\}}$$

integral in polar $\displaystyle\int_0^{\pi/2}\int_{\cos\theta}^{1} (\underbrace{r^2}_{x^2+y^2})^{3/2}\, \overbrace{r\,dr\,d\theta}^{dA}$

$$= \int_0^{\pi/2}\int_{\cos\theta}^{1} r^4\,dr\,d\theta = \cdots = \boxed{\frac{\pi}{10} - \frac{8}{75}}$$

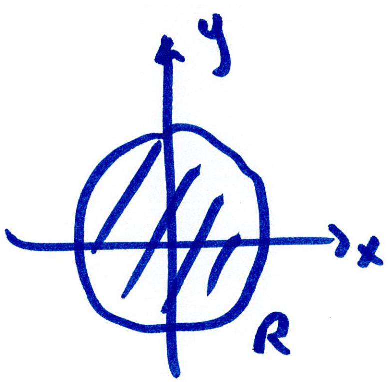
example Find the volume of the solid bounded above by $\underbrace{z=4-x^2-y^2}_{\text{paraboloid}}$ and below by $\underbrace{z=3}_{\text{plane}}$



integrate over R which is the "shadow" of the solid on xy-plane

circle radius ? intersection of $z=3$ and $z=4-x^2-y^2$

$$3=4-x^2-y^2$$
$$x^2+y^2=1$$



bounds: $0 \le \theta \le 2\pi$

$0 \le r \le 1$

$$\text{integrate } \overset{\text{paraboloid}}{\text{top}} - \overset{\text{plane } z=3}{\text{bottom}} \text{ over } R$$

$$(4-x^2-y^2)-(3)$$

$$= 4-r^2-3 = 1-r^2$$

$$\int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta = \cdots = \boxed{\frac{\pi}{2}}$$