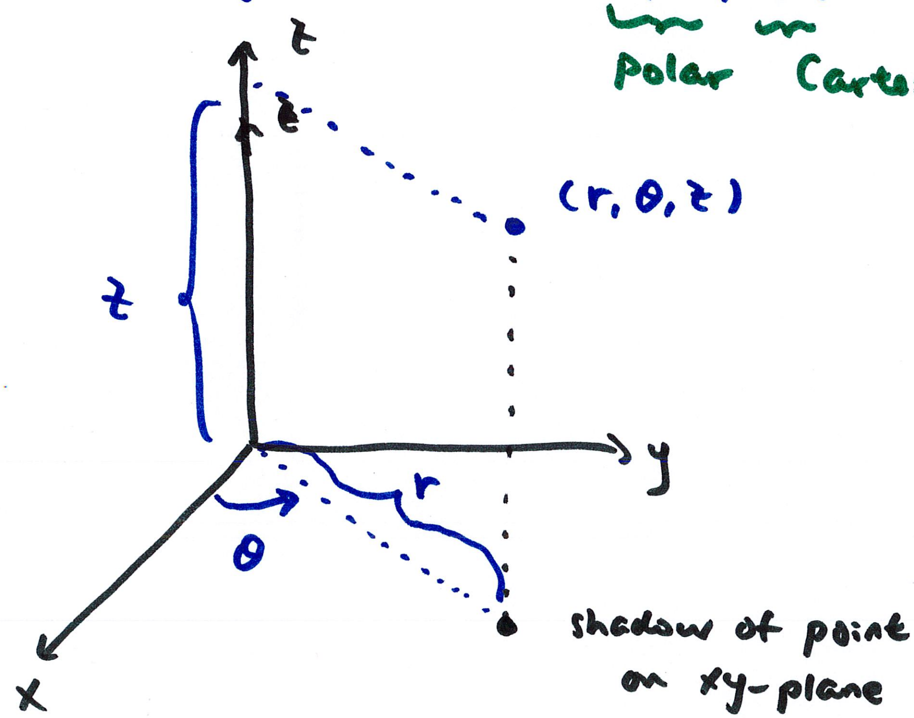
16.5 Triple Integrals in Cylindrical Coordinates

cylindrical: hybrid of polar and cartesian

plane of "floor" → polar

height → Cartesian

point in cylindrical : $(\underbrace{r, \theta}_{\text{Polar}}, \underbrace{z}_{\text{Cartesian}})$



conversion: $(x, y, z) \to (r, \theta, z)$

$$\left.\begin{array}{l} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{array}\right\} \text{polar}$$

$z = z$

$(r, \theta, z) \to (x, y, z)$

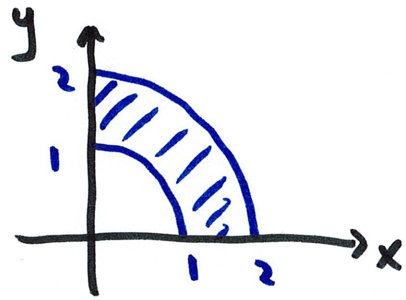
$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \text{polar}$$

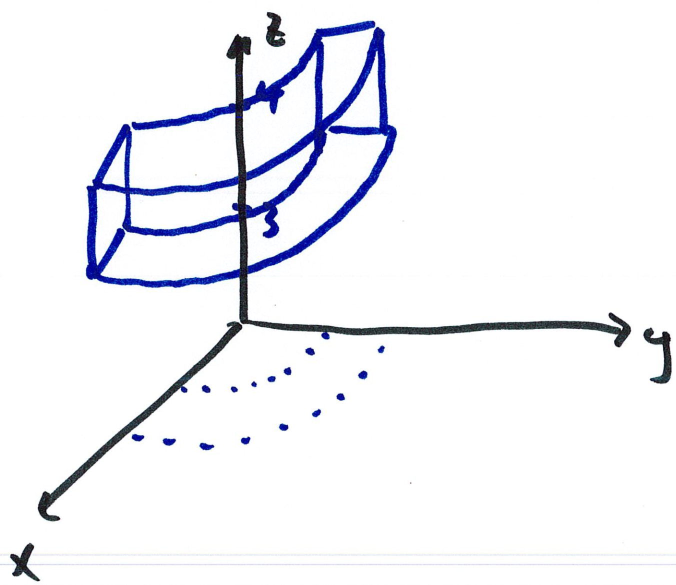
$z = z$

cylindrical is good when volume is cylinder-like

cylinder-like volume looks simple in cylindrical

$$\left\{ (r, \theta, z) : \underbrace{1 \le r \le 2, \quad 0 \le \theta \le \pi/2,}_{\text{"floor" in polar}} \quad 3 \le z \le 4 \right\}$$





example

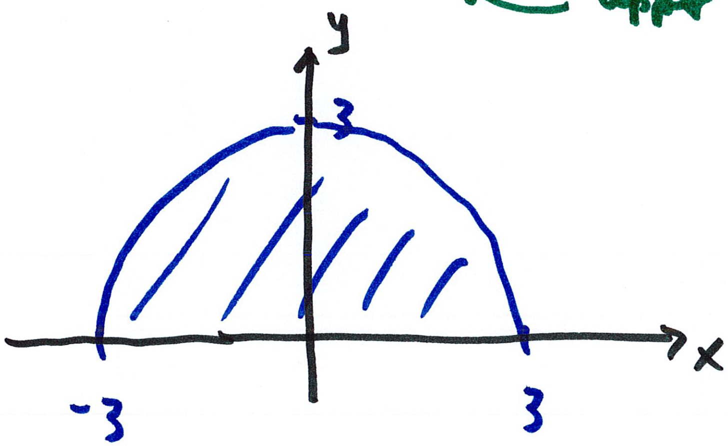
$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$$

terrible in Cartesian

let's examine the volume we are integrating over

$$\left.\begin{aligned}-3 \le x \le 3\\ 0 \le y \le \sqrt{9-x^2}\end{aligned}\right\}\text{ "floor"}$$

← upper half circle radius 3



this is a region in which polar is good

$0 \le \theta \le \pi$

$0 \le r \le 3$

now we look at z

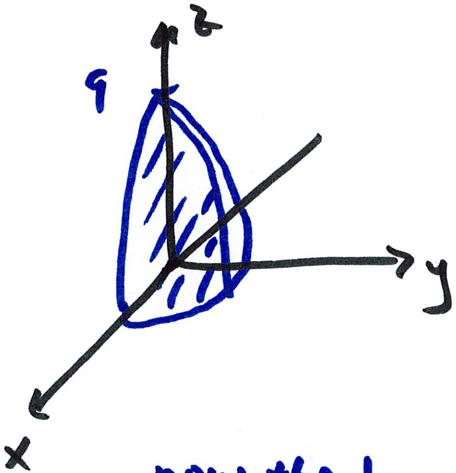
$0 \le z \le \underbrace{9-x^2-y^2}$

xy-plane

paraboloid opening down
vertex at $z=9$

in polar/cylindrical

$$9-x^2-y^2 = 9-(x^2+y^2) = 9-r^2$$



now the bounds in cylindrical: $0 \le r \le 3$, $0 \le \theta \le 2\pi$, $0 \le z \le 9-r^2$

original integral : $$\underbrace{\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}}_{\text{go into } r,\theta \text{ bounds}} \int_0^{\boxed{9-x^2-y^2} \to 9-r^2} \underbrace{\sqrt{x^2+y^2}}_{r}\ \underbrace{dz\,dy\,dx}_{r\,dr\,d\theta}$$

becomes $$\int_{0}^{2\pi}\int_{0}^{3}\int_{0}^{9-r^2} r\ r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^2\,dz\,dr\,d\theta$$

θ r

$$= \cdots = \boxed{\frac{162\pi}{5}}$$

example

$$\int_0^4 \int_0^{1/\sqrt{2}} \int_x^{\sqrt{1-x^2}} e^{-x^2-y^2} \, dy \, dx \, dz$$

can't even start in Cartesian

convert to cylindrical

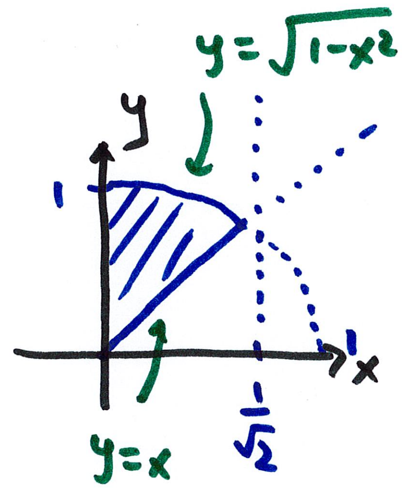
$$0 \le x \le \frac{1}{\sqrt{2}}$$

$$x \le y \le \sqrt{1-x^2}$$

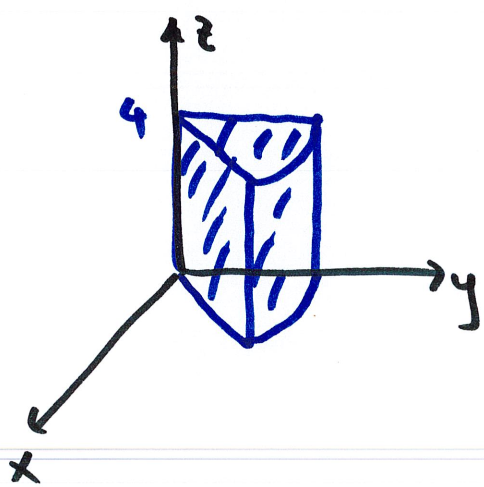
$$0 \le z \le 4$$

line $y=x$

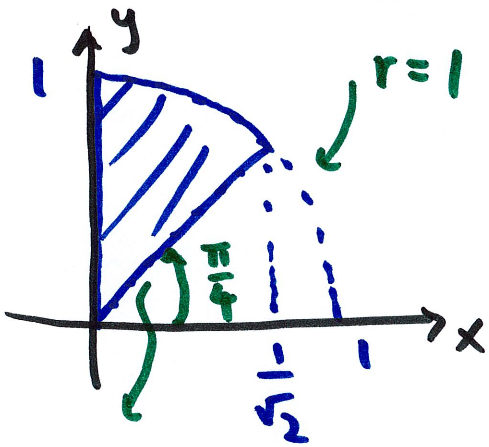
upper circle radius 1



the volume looks like



convert the "floor" to polar



$0 \le r \le 1$

$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$y = x$ slope 1

bisect first quadrant in two halves

so angle is $45^\circ = \frac{\pi}{4}$

convert $$\int_0^4 \underbrace{\int_0^{1/\sqrt{2}} \int_x^{\sqrt{1-x^2}}}_{\text{go to polar}} e^{-x^2-y^2} \underbrace{dy\,dx}_{r\,dr\,d\theta}\,dz \qquad e^{-x^2-y^2} \to e^{-r^2}$$

$0 \le r \le 1$

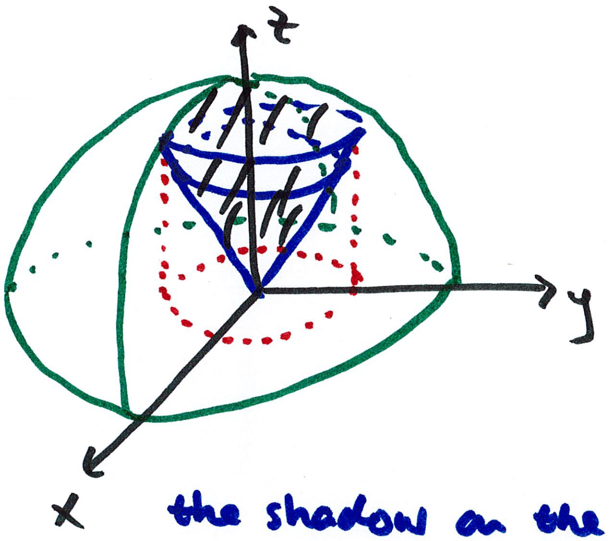
$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

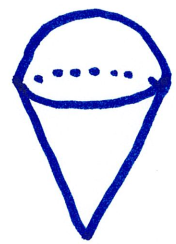
$$= \int_{z=0}^{4} \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{r=0}^{1} r e^{-r^2}\,dr\,d\theta\,dz = \dots = \boxed{\frac{\pi}{2}\left(1-e^{-1}\right)}$$

example Find mass of solid bounded above by $\underbrace{x^2+y^2+z^2=4}_{\text{sphere radius 2}}$

and bounded below by $\underbrace{z=\sqrt{x^2+y^2}}_{\text{cone}}$

with density $\rho(x,y,z)=z$





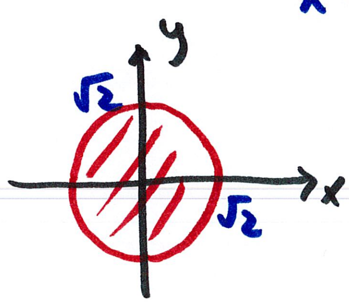
made of material w/ density $= z$

the shadow on the xy-plane is the "floor"

it's a circle with a certain radius

sub $z=\sqrt{x^2+y^2}$ into $x^2+y^2+z^2=4$

$$x^2+y^2+x^2+y^2=4 \rightarrow x^2+y^2=2 \quad \text{circle radius } \sqrt{2}$$



$$0 \le r \le \sqrt{2}$$
$$0 \le \theta \le 2\pi$$

z: above cone $z=\sqrt{x^2+y^2} \rightarrow z=r$

below $z=\sqrt{4-x^2-y^2} \rightarrow z=\sqrt{4-r^2}$

$$r \le z \le \sqrt{4-r^2}$$

$$\text{mass} = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} z \; \underline{\underline{r}}\,dz\,dr\,d\theta = \dots = \boxed{2\pi}$$

we accumulate this (density) inside the icecream cone