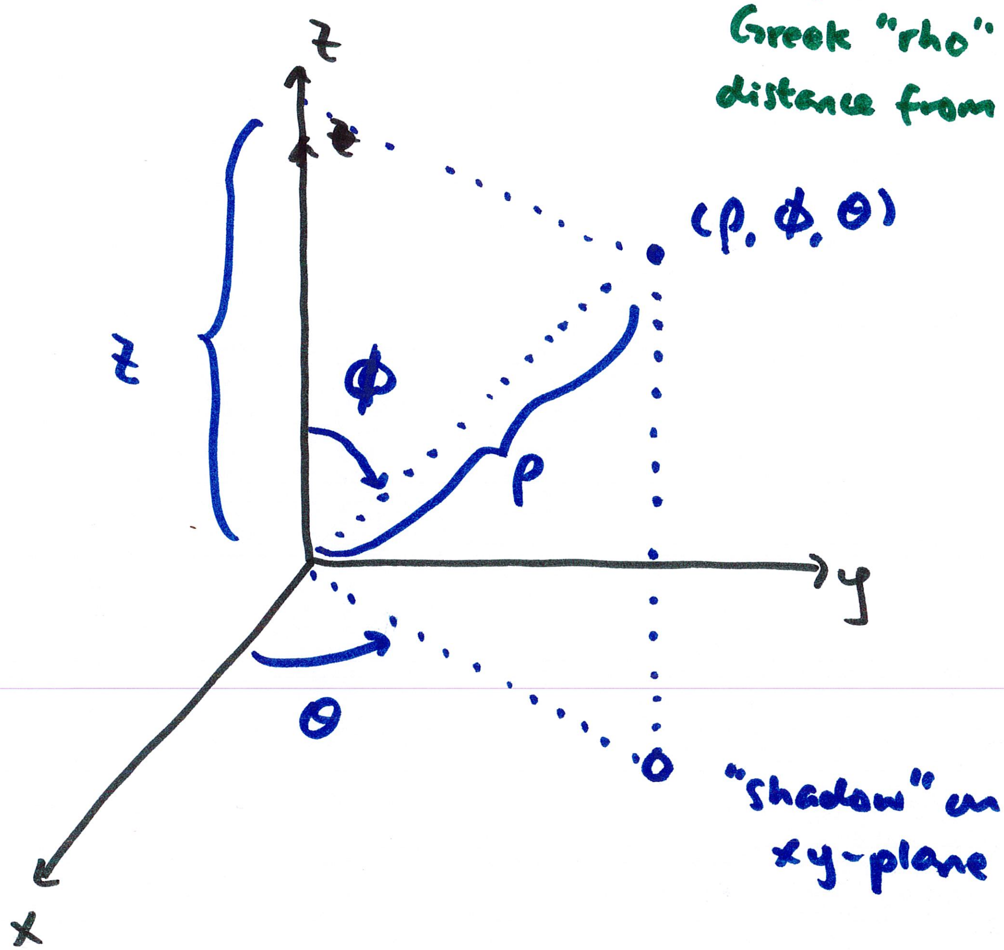
16.5 Triple Integrals in Spherical Coordinates

in spherical, we locate a point by (ρ, ϕ, θ)

Greek "phi" — angle measured from z-axis down

the same θ as in polar or cylindrical

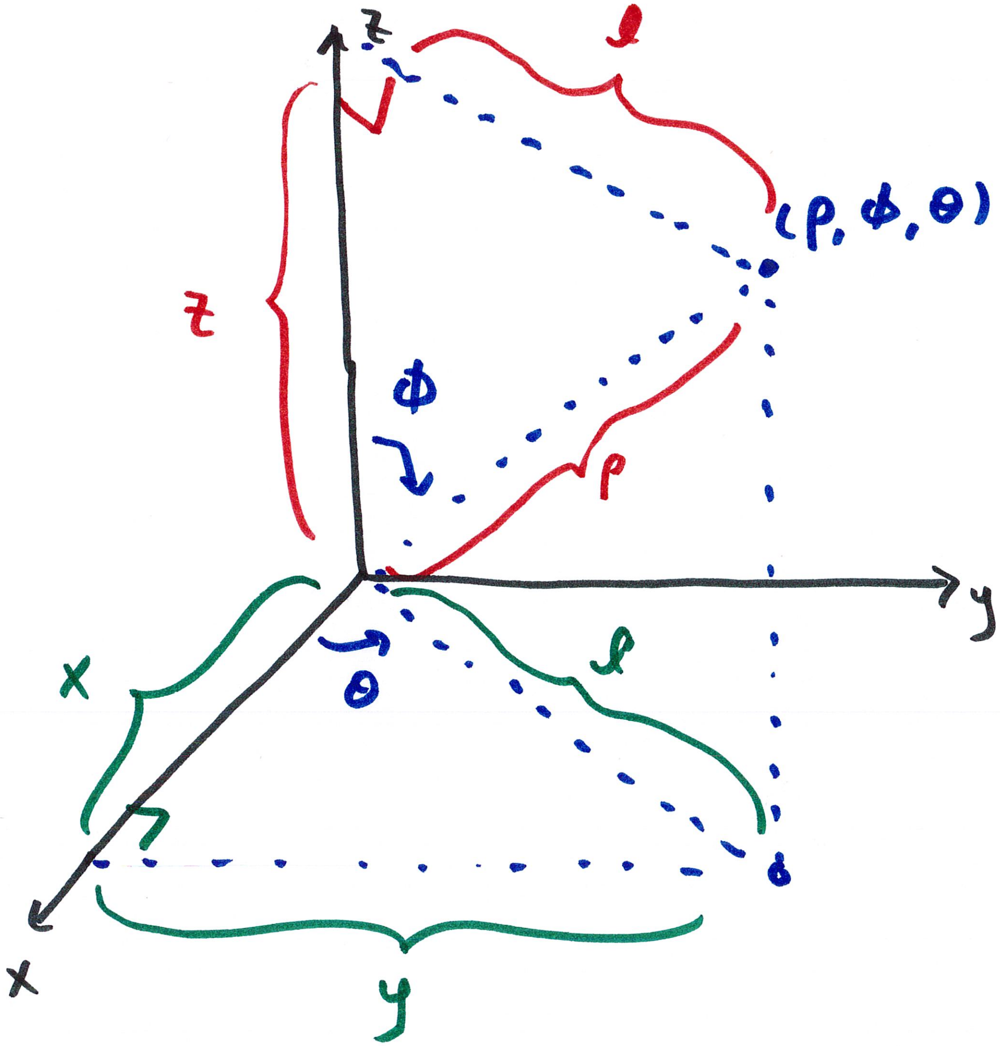
Greek "rho" — distance from origin to point



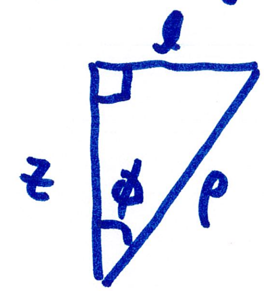
$$\boxed{\begin{array}{c} \rho \geq 0 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi \end{array}}$$

doesn't need to go to 2π

converting from/to Cartesian



top triangle



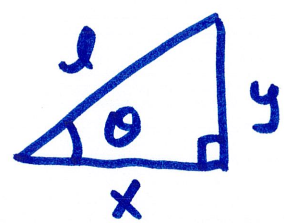
$\cos\phi = \frac{z}{\rho}$

~~$\rho = z\cos\phi$~~

$\boxed{z = \rho\cos\phi}$

$\sin\phi = \frac{\ell}{\rho}$ so $\ell = \rho\sin\phi$

lower triangle



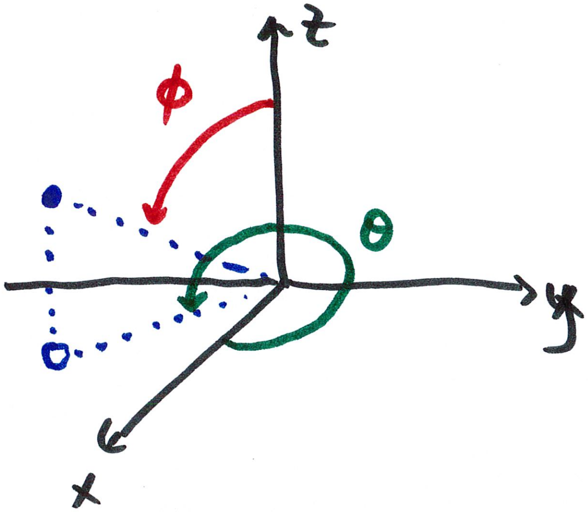
$\cos\theta = \frac{x}{\ell}$ $x = \ell\cos\theta$ $\boxed{x = \rho\sin\phi\cos\theta}$

$\sin\theta = \frac{y}{\ell}$ $y = \ell\sin\theta$ $\boxed{y = \rho\sin\phi\sin\theta}$

note : $\boxed{x^2 + y^2 + z^2 = \rho^2}$

<u>example</u> (Cartesian → spherical)

$$(x, y, z) = (1, -1, \sqrt{2}) \qquad (\rho, \phi, \theta) = ?$$



so, $\frac{3\pi}{2} \le \theta \le 2\pi$

$0 \le \phi \le \frac{\pi}{2}$

ρ: $x^2 + y^2 + z^2 = \rho^2$

$1 + 1 + 2 = \rho^2$ so, $\boxed{\rho = 2}$

ϕ: $z = \rho \cos\phi$

$\sqrt{2} = 2\cos\phi$ $\quad \cos\phi = \frac{\sqrt{2}}{2}$ $\rightarrow$ $\boxed{\phi = \frac{\pi}{4}}$

θ: $\left.\begin{array}{l} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \end{array}\right\}$ $\frac{y}{x} = \tan\theta = \frac{-1}{1} = -1$

QIV θ such that $\tan\theta = -1$

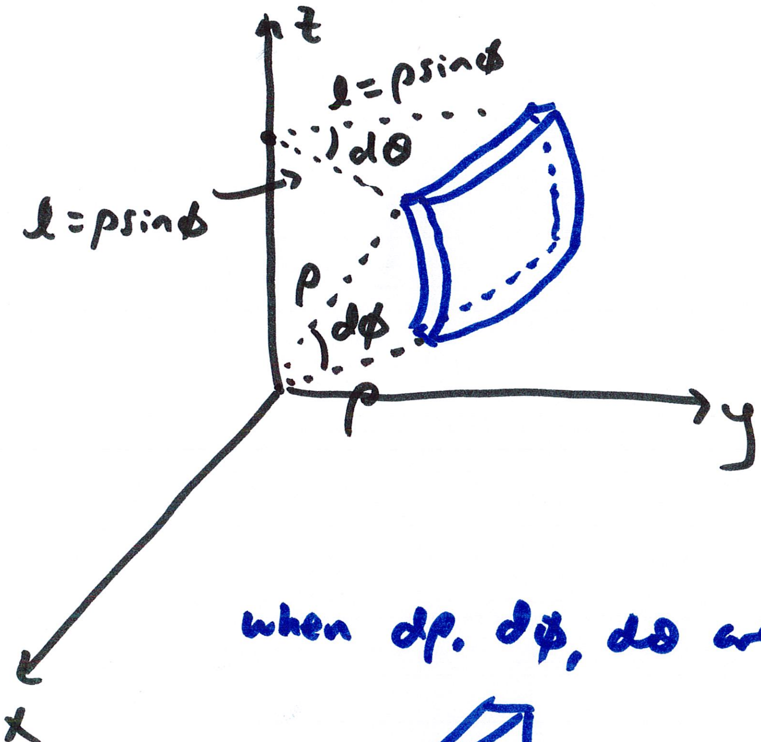
$\boxed{\theta = \frac{7\pi}{4}}$

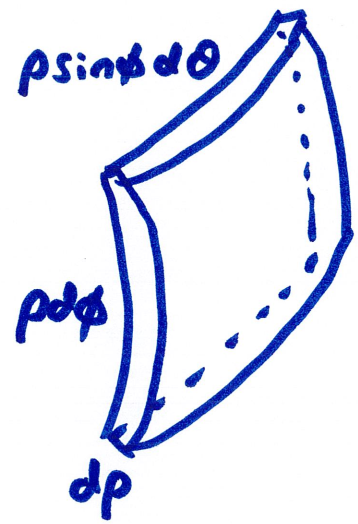
in triple integral, $dV = dz\,dy\,dx$ in Cartesian

$dV = r\,dz\,dr\,d\theta$ in Cylindrical

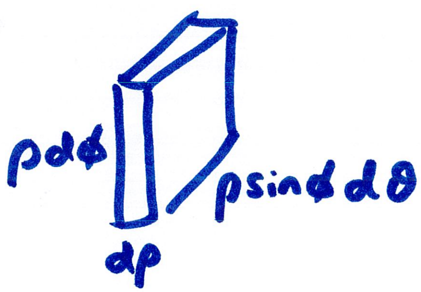
$dV = ?$ Spherical

take a small spherical shell from small changes in ρ, θ, ϕ





when $d\rho$, $d\phi$, $d\theta$ are small, shell $\approx$ rectangular box



$$\boxed{dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}$$

example

$$\int_0^6 \int_0^{\sqrt{36-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{72-x^2-y^2}} dz\,dy\,dx$$

terrible in Cartesian

the upper bound of z is part of a sphere: $z = \sqrt{72-x^2-y^2}$

$z^2 = 72 - x^2 + y^2$

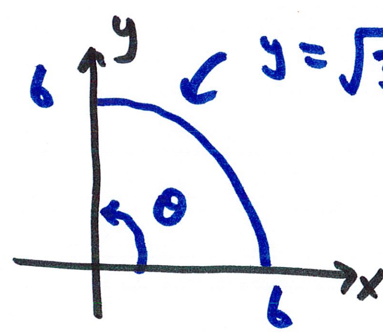
$x^2+y^2+z^2 = 72$

when dealing with spheres or spheres-like, spherical is good

so, go to spherical : bounds for ρ, ϕ, θ ?

$0 \le x \le 6$

$0 \le y \le \sqrt{36-x^2}$

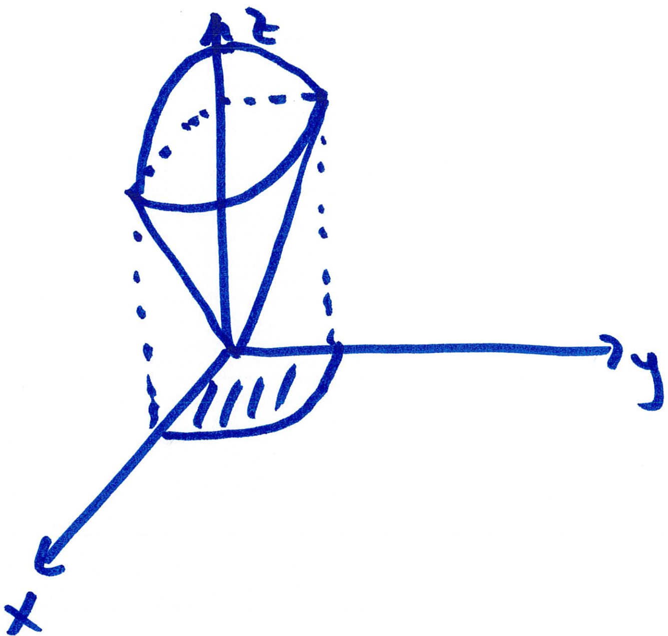


$y = \sqrt{36-x^2}$

this is the shadow onto xy-plane

$$\boxed{0 \le \theta \le \frac{\pi}{2}}$$

$$\underbrace{\sqrt{x^2+y^2}}_{\text{cone}} \le z \le \underbrace{\sqrt{72-x^2-y^2}}_{\text{sphere radius } \sqrt{72}}$$



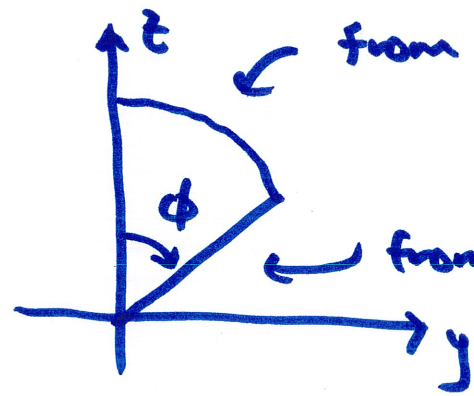
we want everything inside

we accumulate from origin ($\rho = 0$)

out to boundary ($\rho = \sqrt{72}$, sphere)

$$\boxed{0 \le \rho \le \sqrt{72}}$$

onto the yz-plane



ϕ from 0 to where edge of cone is

cone: $z = \sqrt{x^2+y^2} = \sqrt{y^2} = y$

slope 1 → angle is $\frac{\pi}{4}$

so,

$$\boxed{0 \le \phi \le \frac{\pi}{4}}$$

volume is spherical:

$$\int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=0}^{\frac{\pi}{4}} \int_{\rho=0}^{\sqrt{72}} \overbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}^{dV} = \cdots = \boxed{12\sqrt{72}\,\pi\left(1-\frac{1}{\sqrt{2}}\right)}$$

example Volume of solid outside $\underbrace{\rho = 1}_{\substack{\text{sphere radius 1} \\ \text{center } (0,0,0)}}$ and inside $\underbrace{\rho = 2\cos\phi}_{?}$

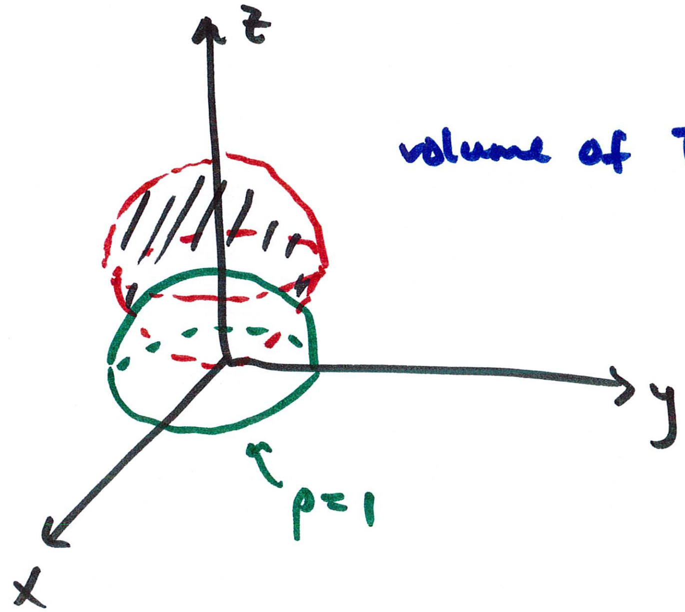
$\rho = 2\cos\phi$

$$\sqrt{x^2+y^2+z^2} = 2\cos\phi \cdot \frac{\rho}{\rho} = \frac{2\overbrace{\rho\cos\phi}^{z}}{\rho} = \frac{2z}{\sqrt{x^2+y^2+z^2}}$$

$x^2+y^2+z^2 = 2z \longrightarrow$

$x^2+y^2+(z-1)^2 = 1$

sphere center $(0,0,1)$ radius 1

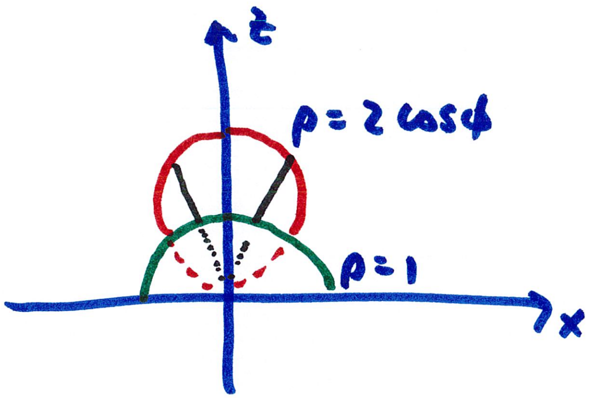


volume of Toad's hat!

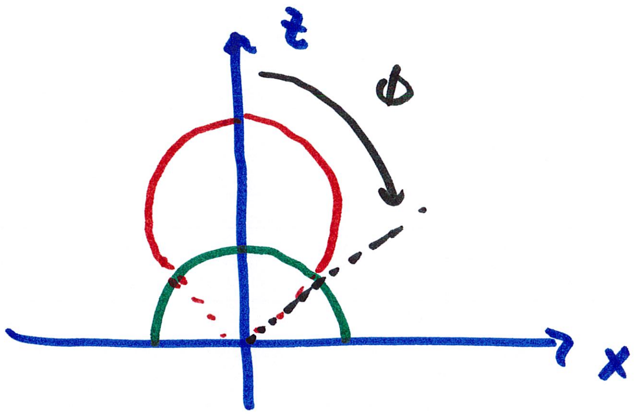
$\boxed{0 \le \theta \le 2\pi}$

all the way around z-axis

xz-plane projection



$$\boxed{1 \le \rho \le 2\cos\phi}$$



ϕ: from z-axis ($\phi = 0$)

to intersection

$\rho = 1$ $\quad \rho = 2\cos\phi$

$1 = 2\cos\phi$ $\quad \cos\phi = \frac{1}{2} \to \phi = \frac{\pi}{3}$

$$\boxed{0 \le \phi \le \frac{\pi}{3}}$$

$$\text{volume} = \int_{0}^{2\pi}\int_{0}^{\pi/3}\int_{1}^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \cdots = \boxed{\frac{11\pi}{2}}$$

$\theta \quad \phi \quad \rho$

the volume of my hat is $\frac{11\pi}{2}$!