17.5 Curl and Divergence

gradient : $\vec{\nabla} f = \underbrace{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle}_{\text{vector}}$

Scalar

must be vector-like

$\vec{\nabla}$ is called the "del operator" defined as

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

in 2D, $\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}$

$\vec{\nabla}$ is a vector-like operator, but it means nothing until applied to a mathematical object (like sin, cos)

gradient : $\vec{\nabla} f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

$\vec{\nabla}$ is like a vector, so we can dot and cross it with another vector

curl $\vec{F}$ is defined as $\vec{\nabla} \times \vec{F}$

$$\boxed{\text{curl } \vec{F} = \vec{\nabla} \times \vec{F}}$$

last time we defined curl $\vec{F}$ when $\vec{F}$ is 2D

$\vec{F} = \langle f, g, 0 \rangle$

from last time

curl $\vec{F} = \langle 0, 0, g_x - f_y \rangle$

verify with the more general formula: $\vec{\nabla} \times \vec{F}$

$$\vec{\nabla} = \cancel{\langle\langle \partial f} \ \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \qquad \vec{F} = \langle f, g, 0 \rangle$$

$$\text{curl}\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & 0 \end{vmatrix}$$

$$= \left\langle \overbrace{\frac{\partial 0}{\partial y}}^{0} - \overbrace{\frac{\partial g}{\partial z}}^{0,\ f \text{ does not depend on } z}, -\left(\underbrace{\frac{\partial 0}{\partial x}}_{0} - \underbrace{\frac{\partial f}{\partial z}}_{0,\ f \text{ doesn't depend on } z} \right), \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle = \langle 0, 0, g_x - f_y \rangle$$

matches definition from last time

<u>example</u> $\vec{F} = \langle xy^2z^3,\ x^3yz^2,\ x^2y^3z \rangle$

$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = ?$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^3 & x^3yz^2 & x^2y^3z \end{vmatrix}$$

$$= \Big\langle \frac{\partial}{\partial y}(x^2y^3z) - \frac{\partial}{\partial z}(x^3yz^2),\ -\left(\frac{\partial}{\partial x}(x^2y^3z) - \frac{\partial}{\partial z}(xy^2z^3)\right)$$
$$\frac{\partial}{\partial x}(x^3yz^2) - \frac{\partial}{\partial y}(xy^2z^3) \Big\rangle$$

$$= \langle 3x^2y^2z - 2x^3yz,\ 3xy^2z^2 - 2xy^3z,\ 3x^2yz^2 - 2xyz^3 \rangle$$

from a previous lesson → conservative vector field is irrotational

(if $\vec{F}$ is conservative, then $\vec{\nabla} \times \vec{F} = \vec{0}$)

let's prove this statement

if $\vec{F}$ is conservative, then $\vec{F} = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\rangle = \vec{\nabla} \phi = \text{grad}\, \phi$

$\vec{\nabla} \times (\vec{\nabla} \phi) = \text{curl}\,(\text{grad}\, \phi)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \left\langle \underbrace{\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}}_{\phi_{zy} - \phi_{yz} = 0}, \; \underbrace{-\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x} \right)}_{0}, \; \underbrace{\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}}_{0} \right\rangle = \vec{0}$$

(mixed partials are equal)

so, this proves the statement

the dot product of $\vec{\nabla}$ with $\vec{F}$ gives us the divergence of $\vec{F}$

$$\boxed{\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}}$$

in 2D, $\vec{F} = \langle f, g, 0\rangle$ $\quad \operatorname{div}\vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \cdot \langle f, g, 0\rangle$

$= f_x + g_y$ (same as how we defined last time)

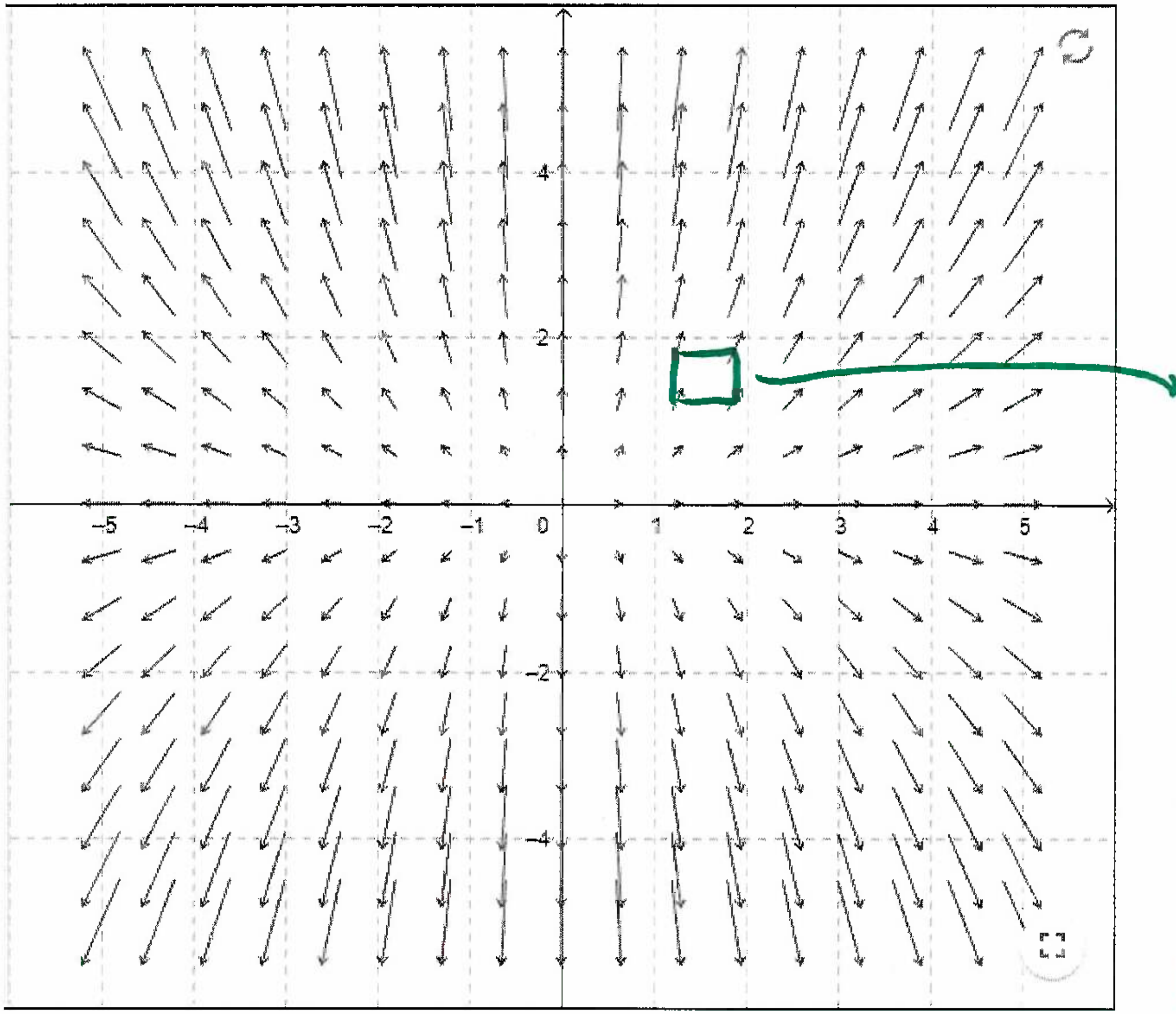
example $\vec{F} = \langle xy^2z^3, x^3yz^2, x^2y^3z\rangle$

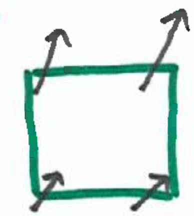
$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \cdot \langle xy^2z^3, x^3yz^2, x^2y^3z\rangle$$

$$= \frac{\partial}{\partial x}(xy^2z^3) + \frac{\partial}{\partial y}(x^3yz^2) + \frac{\partial}{\partial z}(x^2y^3z)$$ note this is a scalar!

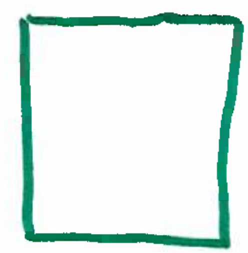
$$= y^2z^3 + x^3z^2 + x^2y^3$$

divergence can be interpreted as the volume change of a small imaginary box that travels w/ the vector field

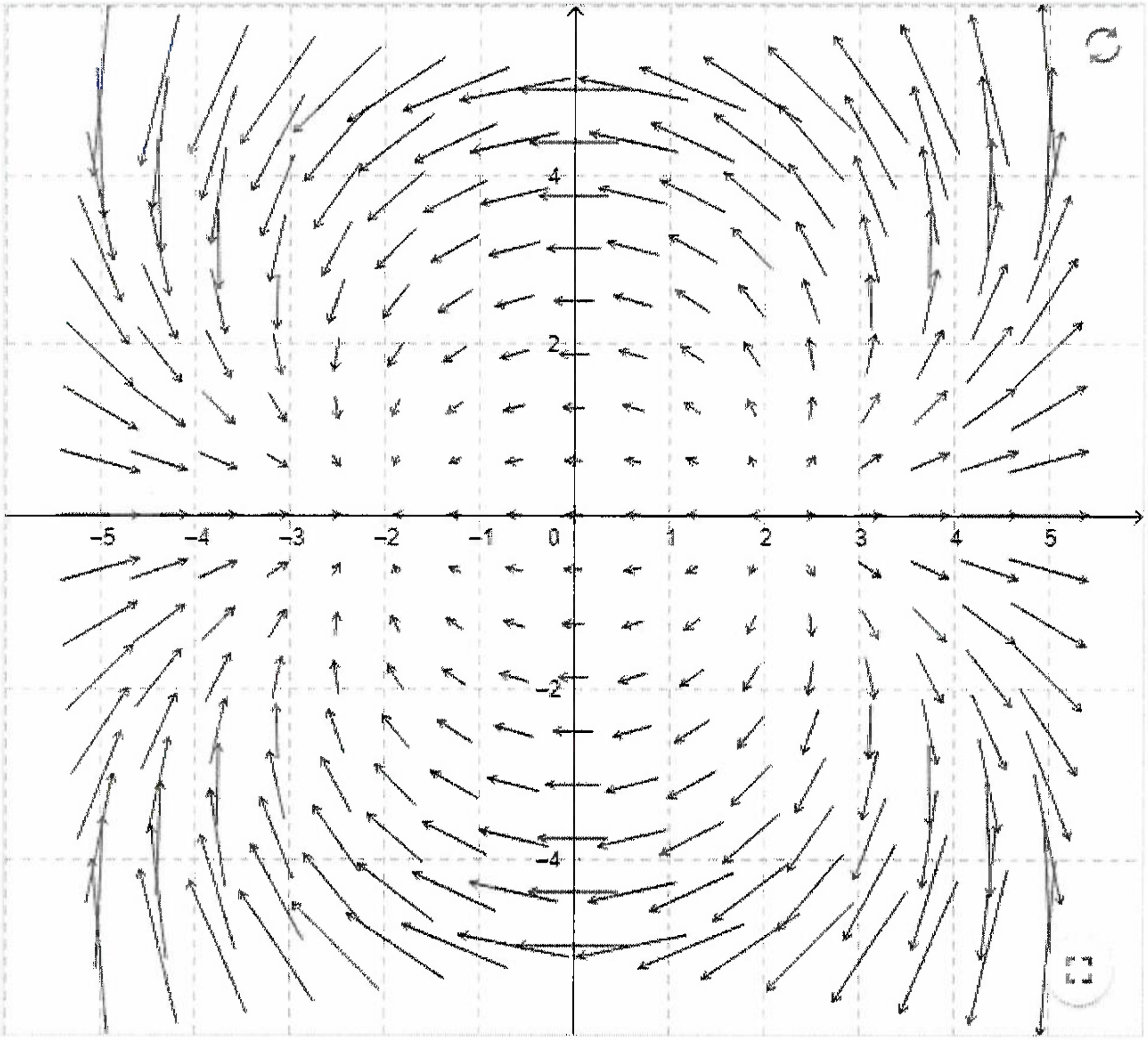




greater y-component pull at top, so as it flows, it will look like



the increase in volume indicates a positive divergence in first quadrant



$\vec{F} = \langle x^2 - y^2 - 4,\ 2xy \rangle$

find curl and divergence

$\text{div}\,\vec{F} = 4x$

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x^2 - y^2 - 4, 2xy, 0 \rangle$$

$$= 2x + 2x = 4x$$

$\text{div}\,\vec{F}$ at a point $(x, y) = 4x$

divergence can have different values at different locations

$\text{curl}\,\vec{F} = \langle 0, 0, 4y \rangle \neq \vec{0}$ all the time, so this $\vec{F}$ is NOT conservative

the rotation strength depends on (x, y), too.