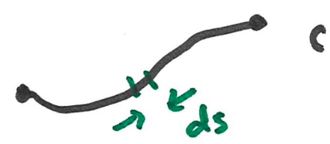
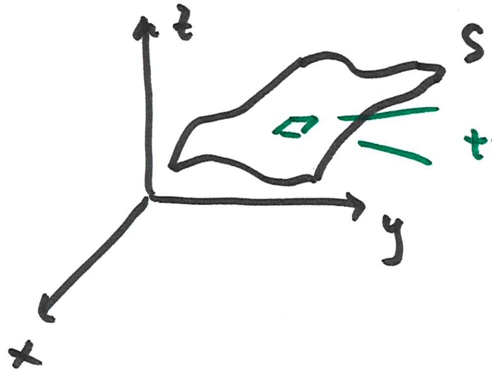
17.6 Surface Integrals (part 1)

line integral: $\int_C f(x,y,z)\,ds$



accumulation of $f(x,y,z)$ along a curve C parametrized

Surface integral: a accumulation of $f(x,y,z)$ all over a surface S (capital S)



tiny patch of the surface S

call this dS

↑ capital S

the total accumulation over this surface looks like

$$\iint_S f(x,y,z)\,dS$$

to compute a line integral $\int_C f\,ds$, we need to parametrize the curve C

example: line from $(-1, 1)$ to $(2, 4)$

$$\vec{r}(t) = \langle -1, 1 \rangle + t \langle 3, 3 \rangle = \langle -1+3t,\ 1+3t \rangle$$

$$0 \le t \le 1$$

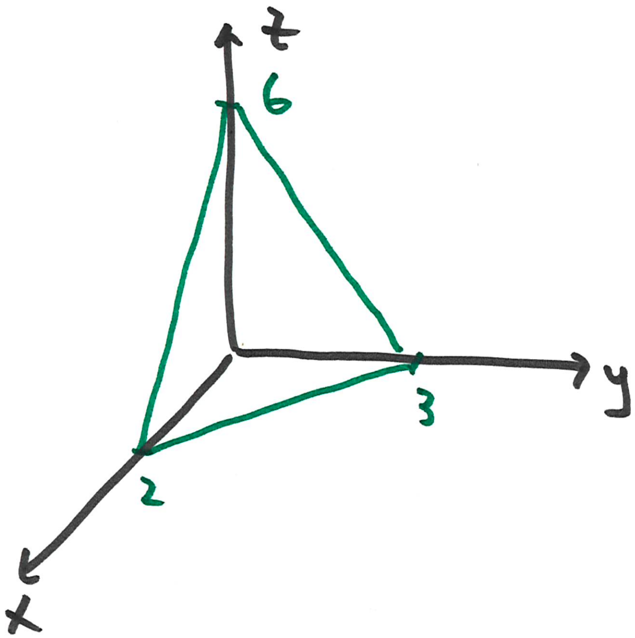
parametrization is not unique

to compute a surface integral $\iint_S f\,dS$, we need to parametrize S

this is one dimension higher than curve, so one more parameter

surface: $\vec{r}(\underbrace{u, v}_{\text{parameters}})$, then the domains of u, v

example Parametrize the part of $3x+2y+z=6$ in first octant



parametrize: how to locate points on the surface

one possible way: specify x, y, then $z = 6-3x-2y$

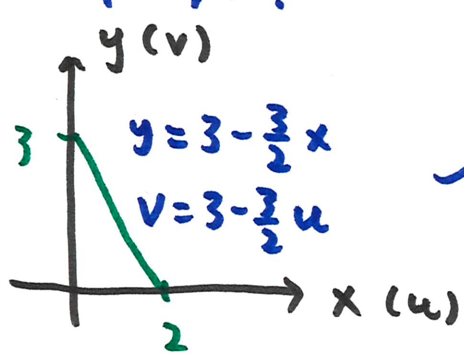
so, we can let $u=x$, $v=y$, then

$$z = f(x,y) = f(u,v) = 6-3u-2v$$

one possible parametrization: $\vec{r}(u,v) = \langle u, v, 6-3u-2v \rangle$

(x y) (x y z)

bounds of u, v?



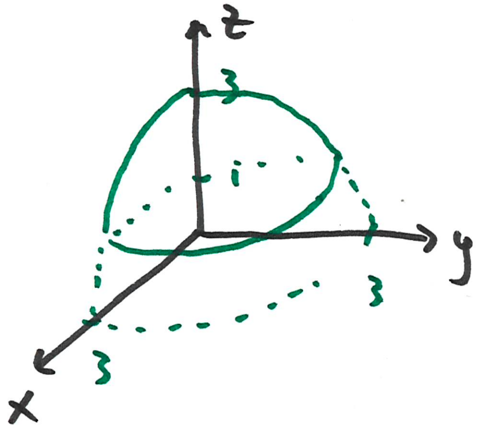
$0 \le u \le 2, \quad 0 \le v \le 3 - \frac{3}{2}u$

parametrization:

$\vec{r}(u,v) = \langle u, v, 6-3u-2v \rangle$

$0 \le u \le 2, \quad 0 \le v \le 3-\frac{3}{2}u$

example $x^2+y^2+z^2=9$ above $z=1$ in first octant

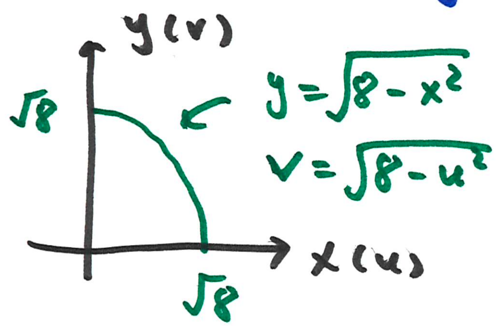


one choice of parametrization

$$x=u, \quad y=v, \quad z=\sqrt{9-x^2-y^2}=\sqrt{9-u^2-v^2}$$

$$\vec{r}(u,v)=\langle u, v, \sqrt{9-u^2-v^2}\rangle$$

at $z=1 \rightarrow x^2+y^2=8$ circle radius $\sqrt{8}$



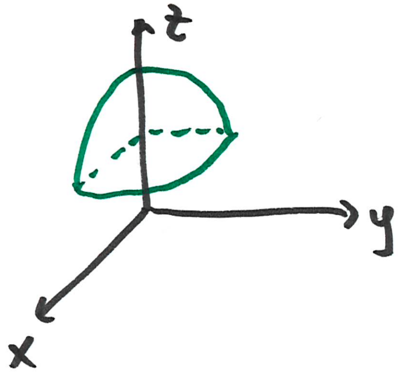
$$0 \le u \le \sqrt{8}, \qquad 0 \le v \le \sqrt{8-u^2}$$

parametrization:

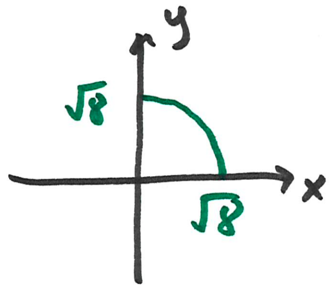
$$\vec{r}(u,v)=\langle u, v, \sqrt{9-u^2-v^2}\rangle$$

$$0 \le u \le \sqrt{8}, \qquad 0 \le v \le \sqrt{8-u^2}$$

Can we do this in another coord. system, for example, cylindrical?



onto xy-plane



$0 \le r \le \sqrt{8}$

$0 \le \theta \le \frac{\pi}{2}$

$1 \le z \le \sqrt{9-r^2}$

choose two to be parameters: $u = r, \quad v = \theta$

then $x = r\cos\theta = u\cos v$ $\qquad 0 \le u \le \sqrt{8}$

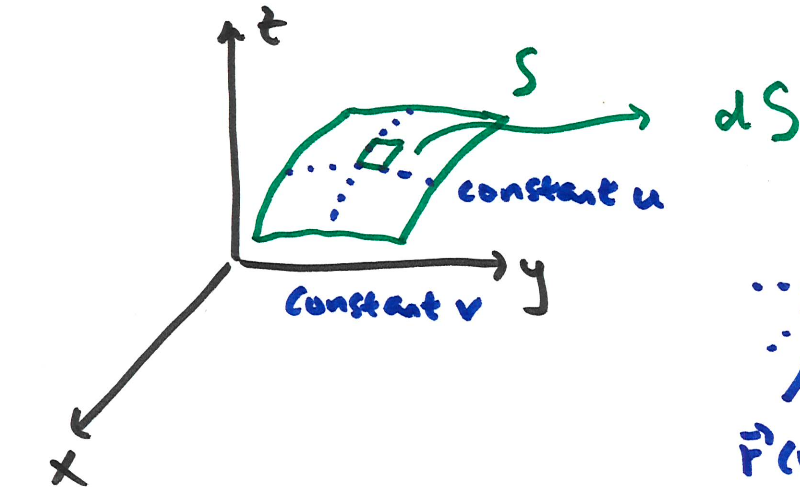
$y = r\sin\theta = u\sin v$ $\qquad 0 \le v \le \frac{\pi}{2}$

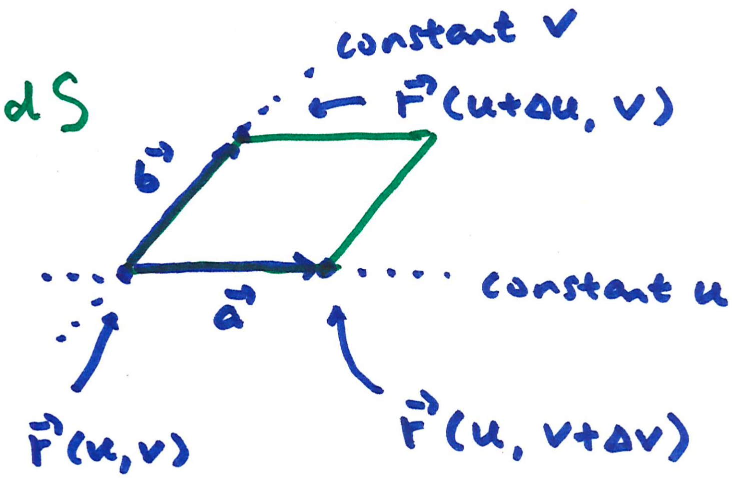
$$\vec{r}(u,v) = \langle \underbrace{u\cos v}_{x = r\cos\theta},\ \underbrace{u\sin v}_{y = r\sin\theta},\ \underbrace{\sqrt{9-u^2}}_{z = z} \rangle$$

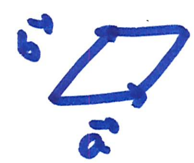
$$0 \le u \le \sqrt{8}, \quad 0 \le v \le \frac{\pi}{2}$$

$\int_C f \, ds$ line integral $ds = |\vec{r}\,'| \, dt$ length of small segment

$\iint_S f \, dS$ surf. integral $dS = ?$ area of small patch of surface







area $= |\vec{a} \times \vec{b}|$

area of dS is $|\vec{a} \times \vec{b}|$

recall: $\dfrac{\partial f}{\partial x} = \lim_{h \to 0} \dfrac{f(x+h, y) - f(x,y)}{h} = f_x$

so, $f_x \approx \dfrac{f(x+\Delta x, y) - f(x,y)}{h}$

or $f(x+\Delta x, y) - f(x,y) \approx f_x h$

now using the above, we get

$$\vec{a} = \vec{r}(u, v+\Delta v) - \vec{r}(u,v) \approx \vec{r}_v \Delta v \qquad \vec{r}_v = \frac{\partial}{\partial v}(\vec{r}(u,v))$$

$$\vec{b} = \vec{r}(u+\Delta u, v) - \vec{r}(u,v) \approx \vec{r}_u \Delta u$$

$$dS = |\vec{a} \times \vec{b}| = |\vec{r}_v \Delta v \times \vec{r}_u \Delta u|$$

$$= |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$

now let $\Delta u, \Delta v \rightarrow du, dv$

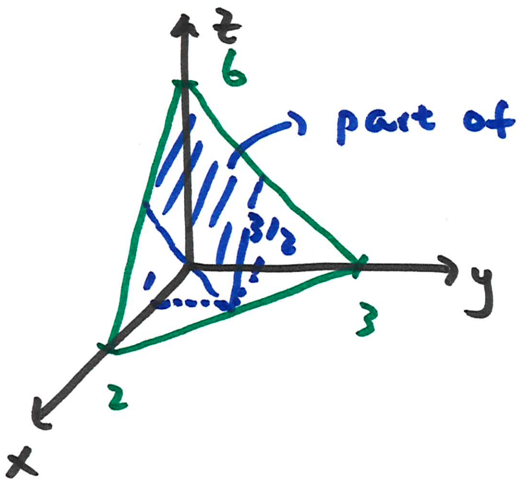
so,

$$\boxed{dS = |\vec{r}_u \times \vec{r}_v| \, du \, dv}$$

this goes in $\iint_S f \, dS$

example $\iint_S (x+y)\,dS$

S: part of plane $3x+2y+z=6$ in the first octant above $0 \le x \le 1$, $0 \le y \le 3/2$



let's reuse parametrization from earlier: $u = x$, $v = y$, $z = 6 - 3u - 2v$

$\vec{r}(u,v) = \langle u, v, 6-3u-2v \rangle \rightarrow \vec{r}_u = \langle 1, 0, -3 \rangle$

$0 \le u \le 1$, $0 \le v \le 3/2$ $\qquad \vec{r}_v = \langle 0, 1, -2 \rangle$

now, $dS = |\vec{r}_u \times \vec{r}_v|\,du\,dv$ $\qquad \vec{r}_u \times \vec{r}_v = \langle 3, 2, 1 \rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{14}$

$dS = \sqrt{14}\,du\,dv$

$$\iint_S (x+y)\, dS$$

$dS \to \sqrt{14}\, du\, dv$

x-component of $\vec{r}(u,v) \to u$

y-component of $\vec{r}(u,v) \to v$

bounds: $0 \le u \le 1$

$0 \le v \le 3/2$

here, any order is ok since both are constant-bounded

$$= \int_{v=0}^{3/2} \int_{u=0}^{1} (u+v)\sqrt{14}\, du\, dv$$

$$= \dots = \boxed{\frac{15\sqrt{14}}{8}}$$

possible interpretation: if $(x+y)$ is density of material, then this is mass