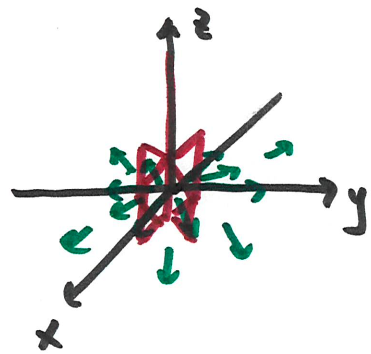
17.7 Stokes' Theorem (part 2)

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$$

let's take another look at the accumulation of curl

curl : $\vec{F} = \langle x, y, 0 \rangle$

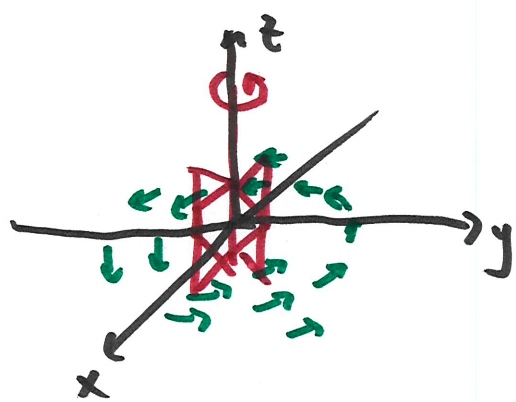


we see no rotation → no curl

if we place a paddle wheel w/ axis along z-axis

here, no rotation is expected

look at $\vec{F} = \langle -y, x, 0 \rangle$

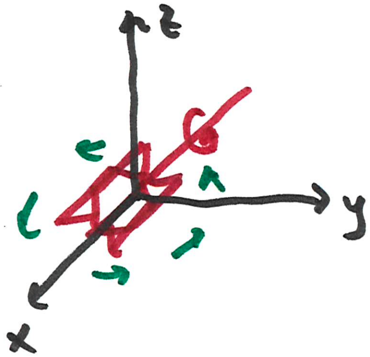


this time, the same paddle wheel will spin

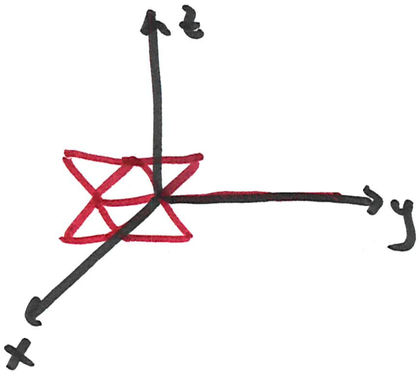
(counterclockwise when viewed

from above)

if the axis of the paddle wheel is tilted



rotation is still expected but not as fast



now no rotation is expected

axis along any coordinate axis results in no rotation

in $\vec{F} = \langle x, y, 0 \rangle$ $\quad$ $\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = \overbrace{\langle 0, 0, 0 \rangle} = \vec{0}$

in $\vec{F} = \langle -y, x, 0 \rangle$ $\quad$ $\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = \underbrace{\langle 0, 0, 2 \rangle}$

this tells us that max. rotation is achieved if paddle wheel axis is along z-axis

$\vec{F} = \langle 5 - z^2, 0, 0 \rangle$

$\text{curl } \vec{F} = \underbrace{\langle 0, -2z, 0 \rangle}$

max rotation achieved wheel axis is along y-axis

as long as part of the wheel's axis is along y-axis → some spin

the component depends on z → speed of rotation depends on z

high z → fast spin

how is this related to Stokes' Theorem?

$$\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \iint_S \text{curl } \vec{F} \cdot \vec{n} \, dS$$

unit normal of surface S

$\text{curl } \vec{F} \cdot \vec{n}$ → projection of $\text{curl } \vec{F}$ onto $\vec{n}$

so, $\text{curl } \vec{F} \cdot \vec{n}$ is seeing how much alignment is between the paddle wheel's spinning axis and the unit normal example

S : upper half of $z^2 = a^2(1-x^2-y^2)$ a : some constant

$\vec{n}$ is positive upward

$\vec{F} = \langle x-y,\ y+z,\ z-x \rangle$

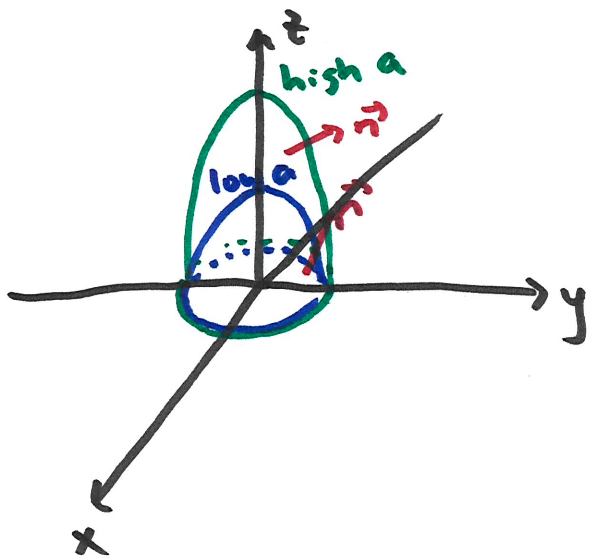
find a such that $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$ is maximized.

shape of $z^2 = a^2(1-x^2-y^2)$?

ellipsoid!

$$z^2 = a^2(1-x^2-y^2)$$

$$\frac{z^2}{a^2} = 1 - x^2 - y^2 \qquad x^2 + y^2 + \frac{z^2}{a^2} = 1$$



a changes how $\vec{n}$ is oriented

let's calculate $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_S \text{curl}\,\vec{F}\cdot \underbrace{(\vec{r_u}\times\vec{r_v})}\,dA$

or $\vec{r_v}\times\vec{r_u}$, whichever points in the positive direction of $\vec{n}$

parametrize surface S.

$z^2 = a^2(1-x^2-y^2)$

upper half: $z = a\sqrt{1-x^2-y^2}$ $(a>0)$

good coordinate system? cylindrical

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

parameters: r, θ, z (choose two)

let $u = r$, $v = \theta$

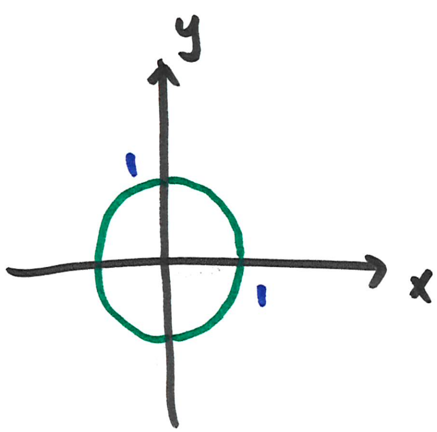
$x = u\cos v$

$y = u\sin v$

$z = a\sqrt{1-u^2}$

$z = a\sqrt{1-(x^2+y^2)} = a\sqrt{1-r^2}$

$= a\sqrt{1-u^2}$



bounds of u, v?

$0 \le u \le 1$

$0 \le v \le 2\pi$

$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ a\sqrt{1-u^2}\rangle$

$$\vec{r}_u = \left\langle \cos v,\ \sin v,\ \frac{-au}{\sqrt{1-u^2}} \right\rangle$$

$$\vec{r}_v = \langle -u\sin v,\ u\cos v,\ 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \left\langle \frac{au^2\cos v}{\sqrt{1-u^2}},\ \frac{au^2\sin v}{\sqrt{1-u^2}},\ u \right\rangle$$

positive? yes.

positive upward → positive z component

z: u, $0 \le u \le 1$

$$\vec{F} = \langle x-y,\ y+z,\ z-x \rangle$$

$$\text{curl}\,\vec{F} = \langle 1, 1, 1 \rangle$$

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_S \text{curl}\,\vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,dA = \int_0^{2\pi}\int_0^1 \langle 1,1,1\rangle \cdot \langle \qquad \rangle\, du\,dv$$

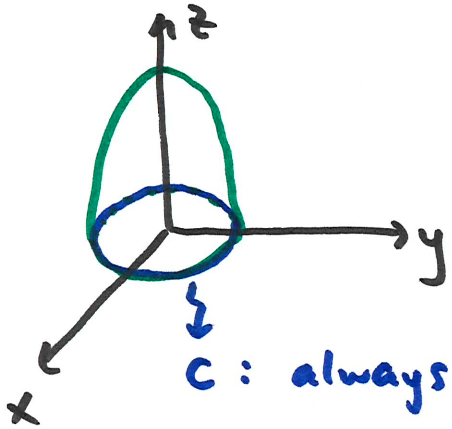
$$= \dots = \boxed{\pi}$$

notice this does not depend on a

$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ does not depend on a

Stokes': $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$

C: boundary curve at the open end of surface



C : always circle radius 1, regardless of a

since C is independent of a, by Stokes' Theorem we expect no dependency of $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ on a

let's work out $\int_C \vec{F}\cdot d\vec{r}$

C : circle radius 1 at $z=0$ $\quad \vec{r}(t) = \langle \underbrace{\cos(t)}_{r\cos\theta}, \sin(t), \underbrace{0}_{xy\text{-plane}} \rangle$

1 → r ← t

$\vec{F} = \langle x-y, y+z, z-x\rangle = \langle \cos t - \sin t, \sin t, -\cos t\rangle$

$d\vec{r} = \vec{r}\,'\,dt = \langle -\sin t, \cos t, 0\rangle\, dt$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle \cos t - \sin t, \sin t, -\cos t\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt$

$= \int_0^{2\pi} -\cos t \sin t + \sin^2 t + \sin t \cos t\, dt = \int_0^{2\pi} \sin^2 t\, dt = \cdots = \boxed{\pi}$