17.8 The Divergence Theorem (part 2)

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_D \operatorname{div} \vec{F} \, dV$$

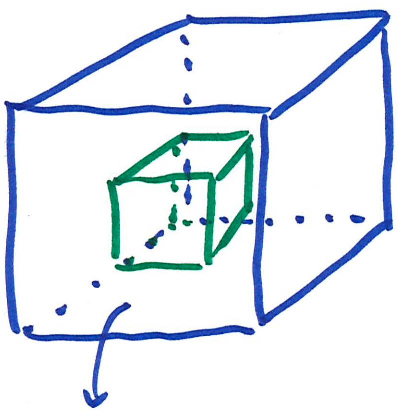
D: space enclosed by S

assumed $\vec{n}$ pointing <u>outward</u>

if $\vec{n}$ is inward, then flip sign

$$-\iiint_D \operatorname{div} \vec{F} \, dV$$

this is useful if the enclosed space has another hollow space



D: space/volume between cubes

remove a smaller cube from inside the big cube

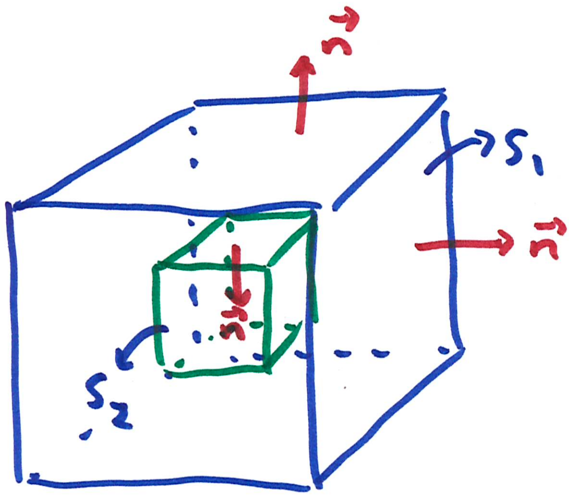
S : surface bounding D is the six outer faces and the six inner faces (that bound the small cube)

S_1 : the outside surface (blue cube)

S_2 : the inside surface (green)

normal vector, as usual, is <u>outward</u> pointing

↳ away from the enclosed volume



on outside, pointing away from volume → out

on inside pointing away from volume → in

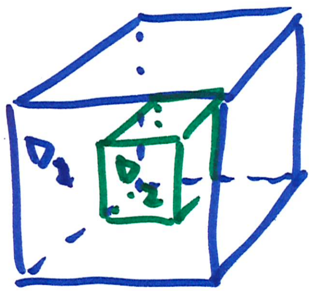
volume ↓ between cubes

the flux integral: $\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_1} \vec{F}\cdot\vec{n}\,dS_1 - \iint_{S_2} \vec{F}\cdot\vec{n}\,dS_2$

$-$ ↓ due to inward normal on inside

the Divergence Theorem then gives

$$\iiint_D \operatorname{div}\vec{F}\,dV = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_1} \vec{F}\cdot\vec{n}\,dS_1 - \iint_{S_2} \vec{F}\cdot\vec{n}\,dS_2$$



D_1 : space inside big cube

D_2 : space inside small cube

apply Divergence Theorem again

$$\iint_{S_1} \vec{F} \cdot \vec{n} \, dS_1 = \iiint_{D_1} \operatorname{div} \vec{F} \, dV$$

$$\iint_{S_2} \vec{F} \cdot \vec{n} \, dS_2 = \iiint_{D_2} \operatorname{div} \vec{F} \, dV$$

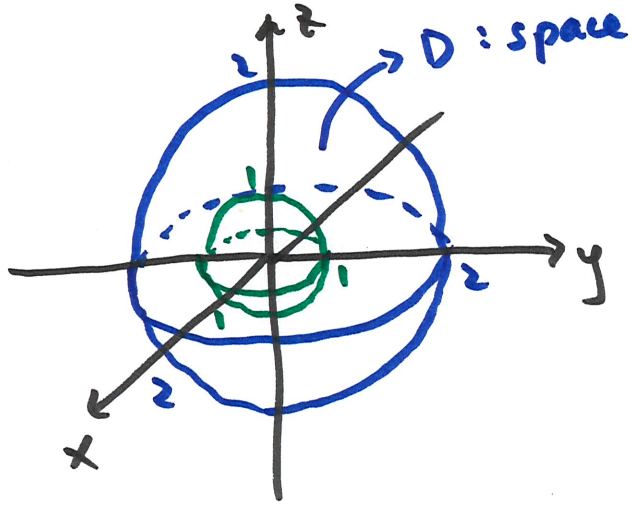
plug those into the equation on previous page, we get

$$\boxed{\iint_{S} \vec{F} \cdot d\vec{S} = \iiint_{D_1} \operatorname{div} \vec{F} \, dV - \iiint_{D_2} \operatorname{div} \vec{F} \, dV}$$

flux of hollow volume : volume integral of div $\vec{F}$ of outer volume

− volume integral of div $\vec{F}$ of inner volume example $\vec{F} = \langle x, y, z \rangle$

D : between spheres radii 2 and 1, both centered at origin
normal is positive outward



D : space between them

choices : $\iint_S \vec{F} \cdot d\vec{S} = \iiint_D \operatorname{div} \vec{F} \, dV$

↳ set up bounds for the space between spheres $1 \le \rho \le 2$, etc

or $\iint_S \vec{F} \cdot d\vec{S} = \underbrace{\iiint_{D_1} \operatorname{div} \vec{F} \, dV}_{\text{big sphere}} - \underbrace{\iiint_{D_2} \operatorname{div} \vec{F} \, dV}_{\text{small sphere}}$

let's try the first way first :

$1 \le \rho \le 2$

$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\langle x,y,z\rangle = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\int_0^{2\pi}\int_0^{\pi}\int_1^2 3\cdot \underbrace{\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}_{dV}$$

$$= 3\int_0^{2\pi}\int_0^{\pi} \frac{1}{3}\rho^3\Big|_1^2 \sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} 7\sin\phi\, d\phi\, d\theta$$

$$= 14\pi\int_0^{\pi}\sin\phi\, d\phi = 14\pi\left(-\cos\phi\right)\Big|_0^{\pi} = \boxed{28\pi}$$

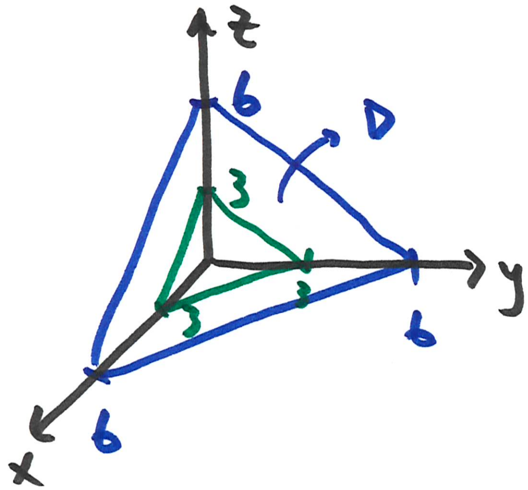
alternative: $\iiint_{D_1}\operatorname{div}\vec{F}\,dV - \iiint_{D_2}\operatorname{div}\vec{F}\,dV$ $\qquad \operatorname{div}\vec{F} = 3$

$$= 3\underbrace{\iiint_{D_1} dV}_{\text{volume of big sphere } \rho=2} - 3\underbrace{\iiint_{D_2} dV}_{\text{volume of small } \rho=1} = 3\left(\frac{4}{3}\pi(2)^3 - \frac{4}{3}\pi(1)^3\right)$$

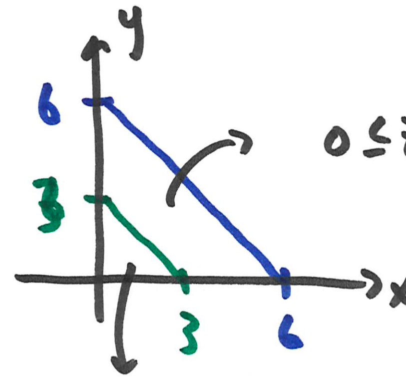
$$= \boxed{28\pi}$$

<u>example</u> $\vec{F} = \langle x^2, -y^2, z^2 \rangle$

D : space bounded by $z = 6 - x - y$ and $z = 3 - x - y$ in first octant



bound for D:



$0 \leq z \leq 6 - x - y$

$3 - x - y \leq z \leq 6 - x - y$

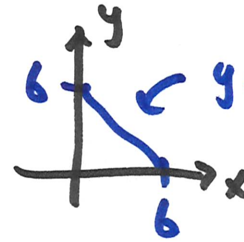
each w/ their own x, y bounds

two integrals needed here, with bounds not simple to set up

alternative: $\iiint\limits_{D_1} \operatorname{div} \vec{F}\, dV - \iiint\limits_{D_2} \operatorname{div} \vec{F}\, dV$

D_1: volume enclosed by higher plane alone

D_2: volume enclosed by lower plane alone the second way seems a little easier



D_1 :

$$0 \le x \le 6$$
$$0 \le y \le 6-x$$
$$0 \le z \le 6-x-y$$

D_2 :

$$0 \le x \le 3$$
$$0 \le y \le 3-x$$
$$0 \le z \le 3-x-y$$

$$\operatorname{div} \vec{F} = 2x-2y+2z$$

minus because D_2 is inner volume

$$\underbrace{\int_0^6 \int_0^{6-x} \int_0^{6-x-y}}_{D_1} (2x-2y+2z)\,dz\,dy\,dx - \underbrace{\int_0^3 \int_0^{3-x} \int_0^{3-x-y}}_{D_2} (2x-2y+2z)\,dz\,dy\,dx$$

$$= \cdots = \boxed{\frac{405}{4}}$$

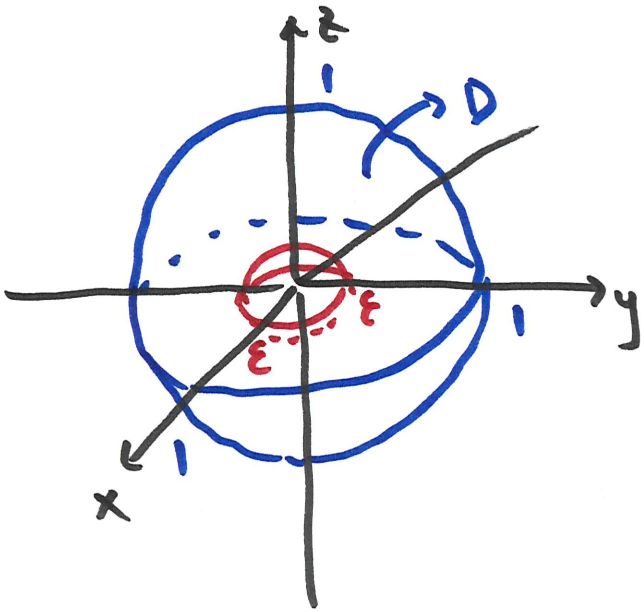
$\iiint_D \operatorname{div}\vec{F}\,dV$ requires $\vec{F}$ to be defined throughout D

if not, then Div. Theorem cannot be used directly

<u>example</u> $\vec{F} = \dfrac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}}$

S : sphere radius 1, centered at origin

at origin, $\vec{F}$ does not exist, so not $\operatorname{div}\vec{F}$, so can't use Div. Theorem directly



work around : make another smaller sphere enclosing origin, then take the limit as its radius → 0

radius = ε , then $\lim_{\varepsilon \to 0}$

in D, $\vec{F}$ exists everywhere

$$\operatorname{div}\vec{F} = \frac{2}{\sqrt{x^2+y^2+z^2}} = \frac{2}{\rho} \text{ in spherical}$$

$$D: \quad \varepsilon \le \rho \le 1$$
$$0 \le \phi \le \pi$$
$$0 \le \theta \le 2\pi$$

$$\iiint_D \operatorname{div}\vec{F}\, dV = \int_0^{2\pi}\int_0^{\pi}\int_\varepsilon^1 \frac{2}{\rho} \overbrace{\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}^{dV}$$

$$= \cdots = 4\pi(1-\varepsilon^2)$$

now $\lim_{\varepsilon\to 0} 4\pi(1-\varepsilon^2) = \boxed{4\pi}$