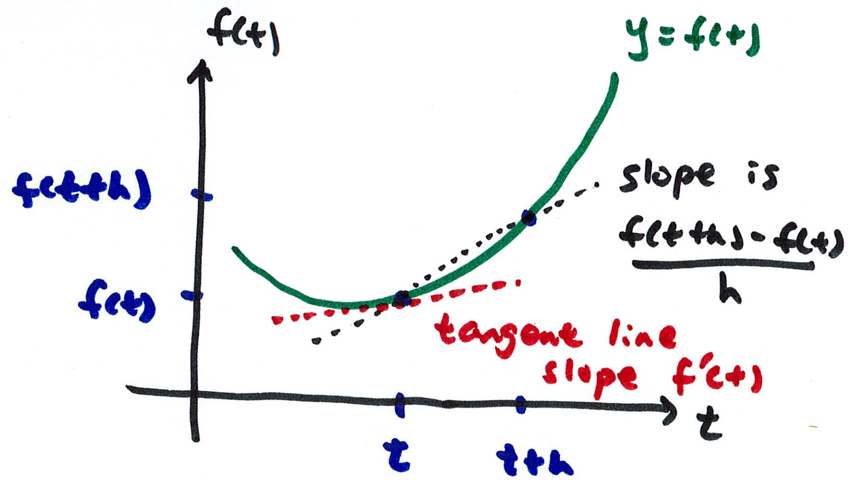
14.2 Calculus of Vector-Valued Functions

recall if $y = f(t)$ is a scalar function

then $y' = f'(t) = \lim\limits_{h \to 0} \dfrac{f(t+h) - f(t)}{h}$



if $\vec{r}(t) = \langle x(t), y(t) \rangle$ is a vector-valued function

its derivative is as expected

$$\boxed{\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}}$$

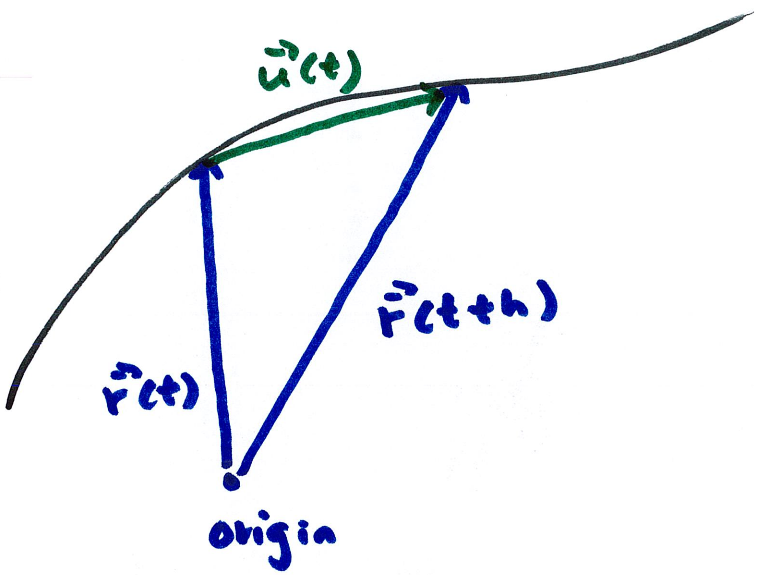
what is the geometric interpretation?

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

then if h is small, then $\vec{r}\,'(t) \approx \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$

or $\vec{r}(t+h) - \vec{r}(t) \approx h\,\vec{r}\,'(t)$

curve from $\vec{r}(t)$



notice $\vec{r}(t) + \vec{u}(t) = \vec{r}(t+h)$

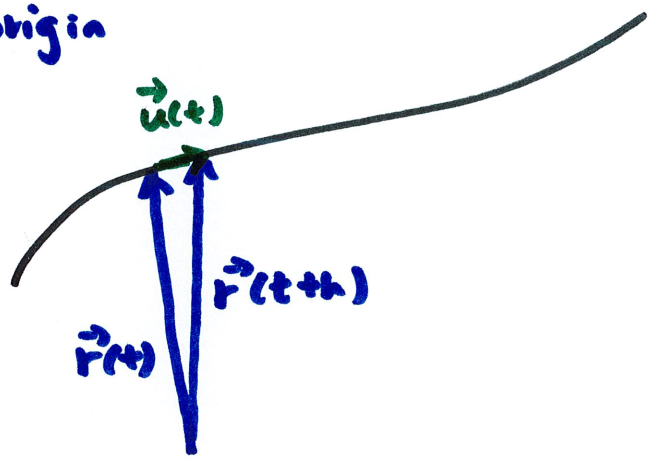
or $\vec{r}(t+h) - \vec{r}(t) = \vec{u}(t)$

as $h \to 0$, h is small

so $\vec{r}(t+h) - \vec{r}(t) \approx h\,\vec{r}\,'(t) = \vec{u}(t)$

$\vec{u}(t)$ will end up following the slope of $\vec{r}(t)$ at $t \to$ <u>tangent vector</u>

as $h \to 0$



$h\vec{r}'(t) = \vec{u}'(t)$ which is tangent vector to $\vec{r}(t)$

so $\vec{r}'(t)$ is also tangent to $\vec{r}(t)$

back to $\vec{r}(t) = \langle x(t), y(t) \rangle$

to calculate $\vec{r}'(t)$, we do $\vec{r}'(t) = \cancel{\lim_{h\to 0} \langle x(t), y(t) \rangle}$

$$= \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$\vec{r}'(t) = \lim_{h\to 0} \left\langle \frac{x(t+h) - x(t)}{h}, \frac{y(t+h) - y(t)}{h} \right\rangle$$

$$= \left\langle \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}, \lim_{h\to 0} \frac{y(t+h) - y(t)}{h} \right\rangle$$

$$\boxed{\vec{r}'(t) = \langle x'(t), y'(t) \rangle}$$

Same is true if $\vec{r}(t)$ has more components example

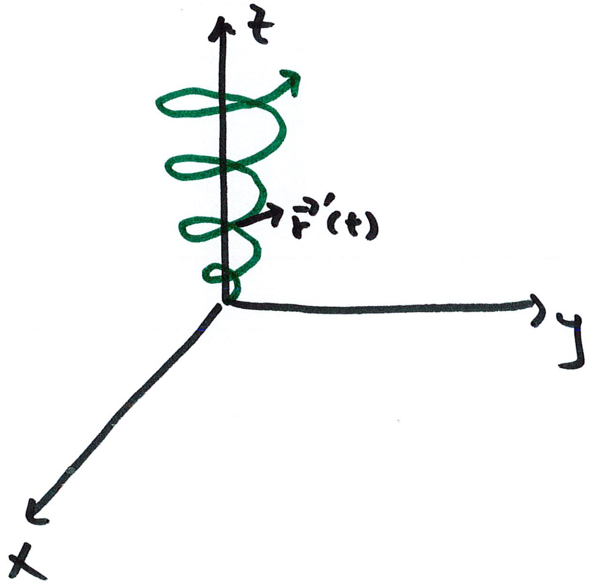
$\vec{r}(t) = \langle t\cos t,\ t\sin t,\ t \rangle$

$\qquad\qquad x \qquad\quad y \qquad z$

$\left.\begin{array}{l} x = t\cos t \\ y = t\sin t \end{array}\right\}\ x^2+y^2 = t^2\cos^2 t + t^2\sin^2 t = t^2(\cos^2 t + \sin^2 t) = t^2$

$z = t$

so, $x^2+y^2 = z^2$ cone z-axis as symmetry axis
cross sections are circles that get bigger as z increases



$\vec{r}(t) = \langle t\cos t,\ t\sin t,\ t \rangle$

$\vec{r}\,'(t) = \langle -t\sin t + \cos t,\ t\cos t + \cancel{\cos t}\ \sin t,\ 1 \rangle$

at $t = \pi/2$

$\vec{r}\,'(\pi/2) = \langle -\frac{\pi}{2}, 1, 1 \rangle$ vector tangent to $\vec{r}(t)$ at $t = \pi/2$

$$|\vec{r}\,'(t)| = \sqrt{(-t\sin t + \cos t)^2 + (t\cos t + \sin t)^2 + (1)^2}$$

$$= \dots = \sqrt{t^2+2}$$ this tells us the magnitude grows as t increases

the tangent vector $\vec{r}\,'(t)$ in general varies in magnitude

later in the course, the unit tangent vector becomes important

unit vector in direction of the tangent vector

$$\boxed{\vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}}$$

for the spiral,

$$\vec{T} = \left\langle \frac{-t\sin t + \cos t}{\sqrt{t^2+2}}, \frac{t\cos t + \sin t}{\sqrt{t^2+2}}, \frac{1}{\sqrt{t^2+2}} \right\rangle$$

many differentiation rules stay the same (or mostly the same)

for example, the product rule for dot product

$$\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$$

$$= \frac{d}{dt}\left[\vec{v}(t)\cdot\vec{u}(t)\right]$$

order doesn't matter for dot product

$$\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \underbrace{\vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)}$$

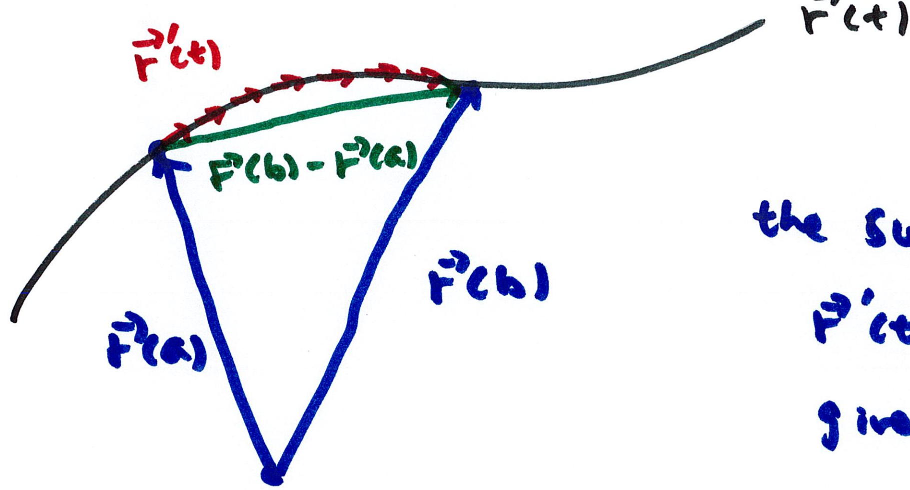
keep the order as written because order is important in cross product integrals work similarly, too

$$\int \vec{r}\,'(t)\,dt = \vec{r}(t) + \vec{C}$$

vector constant of integration

$$\int_a^b \vec{r}\,'(t)\,dt = \vec{r}(b) - \vec{r}(a)$$

this is the displacement vector from $t=a$ to $t=b$



the sum of infinitely many $\vec{r}\,'(t)$ along $t=a$ to $t=b$ gives us $\vec{r}(b)-\vec{r}(a)$

14.3 Motions in space (part 1)

if $\vec{r}(t)$ describes the position of an object

then $\vec{r}\,'(t)$ is the velocity vector

and $|\vec{r}\,'(t)|$ is the speed

and $\vec{r}\,''(t)$ is the acceleration vector

example If $\vec{a}(t) = \langle 1, t, t^2 \rangle$ is the acceleration $(t \geq 0)$

find the velocity such that $\vec{v}(0) = \langle 1, 2, 3 \rangle$

$\vec{a}(t) = \langle 1, t, t^2 \rangle = \vec{v}\,'(t)$

so, $\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle 1, t, t^2 \rangle\, dt$

$$= \left\langle \int 1\,dt,\ \int t\,dt,\ \int t^2\,dt \right\rangle$$

$$= \left\langle t + C_1,\ \tfrac{1}{2}t^2 + C_2,\ \tfrac{1}{3}t^3 + C_3 \right\rangle$$

$$\vec{v}(t) = \left\langle t,\ \tfrac{1}{2}t^2,\ \tfrac{1}{3}t^3 \right\rangle + \underbrace{\langle C_1, C_2, C_3 \rangle}$$

this is the vector constant of integration $\vec{C}$

to find $\vec{C}$, we use $\vec{v}(0) = \langle 1, 2, 3 \rangle$

$$\langle 1, 2, 3 \rangle = \langle 0, 0, 0 \rangle + \langle C_1, C_2, C_3 \rangle$$

so $\vec{v}(t) = \left\langle t,\ \tfrac{1}{2}t^2,\ \tfrac{1}{3}t^3 \right\rangle + \langle 1, 2, 3 \rangle$

$$= \left\langle t+1,\ \tfrac{1}{2}t^2 + 2,\ \tfrac{1}{3}t^3 + 3 \right\rangle$$

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