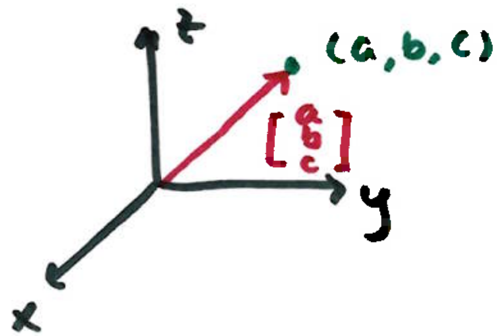
4.1 The Vector Space R^3

R^3 : space containing all points (a, b, c)

each point can be thought of as the tip of a vector whose tail is the origin, then each point is also a 3D vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$



collection of all those vectors is the vector space R^3

(notice R^3 also contains all 2D vectors → xy-plane)

in this vector space all the familiar vector operations hold

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$3\vec{a} + 4\vec{b} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} + \begin{bmatrix} 16 \\ 20 \\ 24 \end{bmatrix} = \begin{bmatrix} 19 \\ 26 \\ 33 \end{bmatrix} = 19\vec{i} + 26\vec{j} + 33\vec{k}$$

$$|\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

two vectors are <u>linearly dependent</u> if one is a scalar multiple of the other: $\vec{u}, \vec{v}$

linearly dependent if $\vec{u} = c\vec{v}$ or $\vec{v} = d\vec{u}$

(parallel to each other)

for example, $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

we see $\vec{v} = -2\vec{u}$ so they are linearly dependent

$\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

we cannot write $\vec{u} = c\vec{v}$

so these vectors are <u>linearly independent</u>

$\boxed{\begin{array}{l}\text{if two vectors } \vec{u} \text{ and } \vec{v} \text{ are linearly independent,} \\ \text{then } \underbrace{a\vec{u} + b\vec{v}}_{\text{linear combination}} = \vec{0} \text{ is possible if and only if } a = b = 0\end{array}}$

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$a\vec{u} + b\vec{v} = \vec{0} \rightarrow a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow A\vec{x} = \vec{b}$$

solve for $\begin{bmatrix} a \\ b \end{bmatrix}$: $\vec{x} = A^{-1}\vec{b}$ $\rightarrow$ unique solution only if A^{-1} exists

$\rightarrow$ $\boxed{\det A \neq 0}$

or Gaussian elimination

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 0 \end{bmatrix}$$

$-4b = 0 \rightarrow b = 0$

$a + 2b = 0$

$a = -2b = -2(0) = 0$

unique solution

$a = b = 0$

so, $\vec{u}, \vec{v}$ are linearly independent

$\vec{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 7 \\ 8 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Can we express one as a linear combination of the other two?

for example, $\vec{u}$ as a linear combination of $\vec{v}$ and $\vec{w}$?

↳ $\vec{u} = a\vec{v} + b\vec{w}$ → find a, b

$$a \begin{bmatrix} 7 \\ 8 \end{bmatrix} + b \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

as matrix equation: $\begin{bmatrix} 7 & -1 \\ 8 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 8 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

or, $\begin{bmatrix} 7 & -1 & 4 \\ 8 & 2 & 3 \end{bmatrix}$ then reduction

$$A = \begin{bmatrix} 7 & -1 \\ 8 & 2 \end{bmatrix} \qquad A^{-1} = \frac{1}{7\cdot 2 - 8\cdot(-1)} \begin{bmatrix} 2 & 1 \\ -8 & 7 \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 2 & 1 \\ -8 & 7 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 2 & 1 \\ -8 & 7 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{22} \begin{bmatrix} 11 \\ -11 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\underset{\frac{1}{2}}{a} \begin{bmatrix} 7 \\ 8 \end{bmatrix} + \underset{-\frac{1}{2}}{b} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

everything we've done so far holds for R^3, R^4, R^5, etc example

$$\vec{u} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix}$$

are they linearly independent ?

if not, write one as a linear combination of the other two.

linearly independent if $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$ implies $a=b=c=0$ as the only solution

as matrix equation

$$\begin{array}{ccc} \vec{u} & \vec{v} & \vec{w} \end{array}$$
$$\begin{bmatrix} 4 & -5 & 0 \\ 0 & 1 & -4 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

options: unique solution only if $\begin{bmatrix} 4 & -5 & 0 \\ 0 & 1 & -4 \\ 1 & -1 & -1 \end{bmatrix}^{-1}$ exists

$\rightarrow$ determinant $\neq 0$

reduce the augmented matrix $\begin{bmatrix} 4 & -5 & 0 & 0 \\ 0 & 1 & -4 & 0 \\ 1 & -1 & -1 & 0 \end{bmatrix}$

reduction is better here because we will find a, b, c after reduction

$$\begin{bmatrix} 4 & -5 & 0 & 0 \\ 0 & 1 & -4 & 0 \\ 1 & -1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -4 & 0 \\ 4 & -5 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & -1 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

↳ infinitely many solutions
BUT linearly independent only if $a=b=c=0$ is the ONLY solution

so, $\vec{u}, \vec{v}, \vec{w}$ are NOT linearly independent and any one can be written as linear combo of the others

from reduced matrix : $c = r$ (free, no pivot in column 3)

$b - 4c = 0 \rightarrow b = 4c = 4r$

$a - b - c = 0 \rightarrow a = b + c = 5r$

$a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$ choose $r = 1 \rightarrow a = 5,\ b = 4,\ c = 1$

$5\vec{u} + 4\vec{v} + \vec{w} = \vec{0}$

$$\boxed{\vec{w} = -5\vec{u} - 4\vec{v}}$$

or $$\boxed{\vec{u} = -\frac{4}{5}\vec{v} - \frac{1}{5}\vec{w}}$$