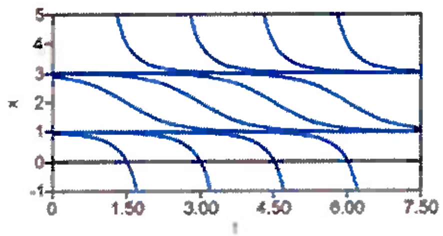
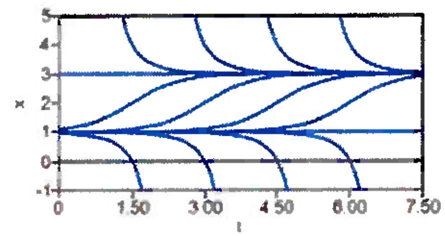
3. Select the graph which shows the solutions of the differential equation

$$x' = x^2 - 4x + 3$$

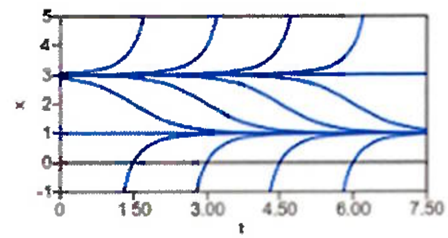
A.



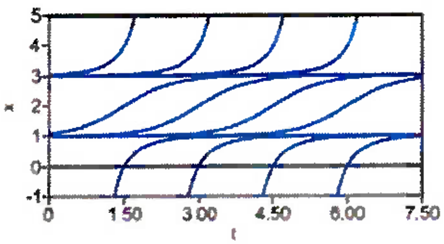
B.



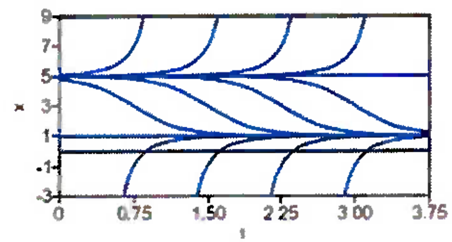
C.



D.



~~*E.*~~



Straight lines: equilibrium solution → $x' = 0$

$x^2 - 4x + 3 = 0$ $\quad (x - 3)(x - 1) = 0$

$x = 1, \quad x = 3$

stable $\quad$ unstable

$x' \rightarrow 0 \leftarrow 0 \rightarrow$

$x=0$, $x'>0$ $\quad$ $x=1$ $\quad$ $x=2$, $x'<0$ $\quad$ $x=3$ $\quad$ $x=4$, $x'>0$

1. Let $y(x)$ be the solution to the initial value problem

$$\underbrace{xy' + y = 3x^2 + 2x - 4}_{} \qquad y(1) = 0.$$

1st-order linear

integrating factor

$$y' + \frac{1}{x} y = 3x + 2 - \frac{4}{x}$$

$$I = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

multiply $y' + \frac{1}{x} y = 3x + 2 - \frac{4}{x}$

by integrating factor on both sides

left: $\frac{d}{dx}(x \cdot y) = 3x^2 + 2x - 4$

integrating factor

Then $y(2) =$

A. 1

B. 2

C. 3

D. 4

E. 5

$$xy = x^3 + x^2 - 4x + C$$

$$y = x^2 + x - 4 + \frac{C}{x}$$

$$y(1) = 0 = -2 + C \qquad C = 2$$

$$y = x^2 + x - 4 + \frac{2}{x} \qquad y(2) = 4 + 2 - 4 + 1 = 3$$

4. A container initially contains 10 L of water in which 20 g of salt is dissolved. A solution containing 4 g/L of salt is pumped into the tank at a rate of 2 L/min and the well stirred mixture runs out of the tank at a rate of 1L/min. What is the concentration of the salt in the tank after 10 minutes?

A. 1.5

B. 2.5

C. 3.5

D. 4.5

E. 5.5

let $y(t)$ be amount of salt in g $\quad y(0)=20$

$$\frac{dy}{dt} = (\text{concentration in})(\text{flow rate in}) - (\text{conc. out})(\text{flow rate out})$$

$$= (4)(2) - \left(\frac{y}{\underbrace{10+t}}\right)(1)$$

net gain of 1 L/min
started w/ 10

$$y' = 8 - \frac{y}{10+t} \quad \text{linear}$$

$$y' + \frac{1}{10+t} y = 8 \qquad I = e^{\int \frac{1}{10+t} dt} = e^{\ln(10+t)} = 10+t$$

$$\frac{d}{dt}\left[(10+t)y\right] = 8(10+t) = 80 + 8t$$

$$(10+t)y = 80t + 4t^2 + C \qquad y(0) = 20$$

$$(10)(20) = C = 200$$

$$y = \frac{80t + 4t^2 + 200}{10+t}$$

$$\text{concentration after 10 minutes} = \frac{\text{amount after } t=10}{\text{volume at } t=10}$$

$$= \frac{\frac{800 + 400 + 200}{20}}{20} = 3.5$$

8. Find a basis for the null space of $A = \begin{bmatrix} 1 & -2 & 2 & 3 \\ 2 & -4 & 5 & 7 \\ 3 & -6 & 3 & 6 \end{bmatrix}$.

A. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

B. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

C. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.

D. $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

E. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

null space : solution space of $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & -2 & 2 & 3 & 0 \\ 2 & -4 & 5 & 7 & 0 \\ 3 & -6 & 3 & 6 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 \end{bmatrix}$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\to \begin{bmatrix} \boxed{1} & -2 & 2 & 3 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

two pivots

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

x_2, x_4 free

$x_2 = a \quad x_4 = b$

row 2: $x_3 + x_4 = 0 \to x_3 = -b$

row 1: $x_1 = 2x_2 - 2x_3 - 3x_4$

$= 2a + 2b - 3b = 2a - b$

$$\vec{x} = \begin{bmatrix} \cancel{2a+2}\ 2a-b \\ a \\ -b \\ b \end{bmatrix} = a\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

two bases → two-dimensional

12. Which of the following are vector spaces?

(i) $\{(x, y) \in \mathbb{R}^2 : \; x > y\}$

(ii) $\{(x, y) \in \mathbb{R}^2 : \; 2x - 3y = 0\}$

(iii) All solutions to $y'' + 2y = 0$ on $(-\infty, \infty)$.

(iv) All solutions to $y'' + 2 = 0$ on $(-\infty, \infty)$.

A. (i), (ii), and (iii)

B. (ii), (iii), and (iv)

C. (ii) and (iii)

D. (i) and (iv)

E. None of them are vector spaces.

Requirements for vector space

1. existence of $\vec{0}$
2. closed under addition
3. closed under scalar multiplication

(i) $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $x > y$ contains $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$?

0 is NOT > 0

so, $\vec{0}$ is NOT in space

NOT a vector space

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ mult. by -1 $\underbrace{\begin{bmatrix} -2 \\ -1 \end{bmatrix}}_{\text{not in space}}$

not closed

(ii) $\begin{bmatrix} x \\ y \end{bmatrix}$ $2x - 3y = 0$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \frac{2}{3}x \end{bmatrix}$

contains $\vec{0}$? yes $(x=0)$

addition? $\begin{bmatrix} x_1 \\ \frac{2}{3}x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ \frac{2}{3}x_2 \end{bmatrix} = \begin{bmatrix} (x_1 + x_2) \\ \frac{2}{3}(x_1 + x_2) \end{bmatrix}$

still in the space

so closed.

mult.? $k \cdot \begin{bmatrix} x_1 \\ \frac{2}{3}x_1 \end{bmatrix} = \begin{bmatrix} (kx_1) \\ \frac{2}{3}(kx_1) \end{bmatrix}$ so is closed

so (ii) is a vector space

(iii) all solutions $y'' + 2y = 0$ on $(-\infty, \infty)$

char. eq. $r^2 + 2 = 0$ $\quad r = \pm\sqrt{2}\, i$

$$y = C_1 \cos(\sqrt{2}t) + C_2 \sin(\sqrt{2}t)$$

is $y = 0$ in here? yes

addition: $y_1 = A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t)$

$$y_2 = E\cos(\sqrt{2}t) + F\sin(\sqrt{2}t)$$

$$y_1 + y_2 = \underbrace{(A+E)}_{C_1}\cos(\sqrt{2}t) + \underbrace{(B+F)}_{C_2}\sin(\sqrt{2}t)$$

closed

mult.: $y_1 = A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t)$

$$Ky_1 = (KA)\cos(\sqrt{2}t) + (KB)\sin(\sqrt{2}t)$$

closed

iv) $y'' + 2 = 0$

$y'' = -2$ is $y = 0$ in it?

$y'' = 0$ but now $y'' \neq -2$ so, no $\vec{0}$

NOT a vector space

19. Find all constants b such that the origin is a spiral source of the system

$$X'(t) = \begin{bmatrix} 3 & b \\ 1 & 4 \end{bmatrix} X(t), \quad b \text{ in } \mathbb{R}$$

are

A. $b < -\dfrac{1}{3}$

B. $b > -\dfrac{1}{3}$

C. $-\dfrac{1}{4} < b < -\dfrac{1}{4}$

D. $b > -\dfrac{1}{4}$

E. $b < -\dfrac{1}{4}$

Spiral source : complex eigenvalues w/ positive real part

$$\begin{vmatrix} 3-\lambda & b \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(4-\lambda) - b = 0$$

$$\lambda^2 - 7\lambda + (12-b) = 0$$

$$\lambda = \frac{7 \pm \sqrt{49 - 4(1)(12-b)}}{2}$$

complex if $49 - 4(12-b) < 0$

$$49-48+4b<0$$

$$1+4b<0 \qquad 4b<-1 \qquad b<-\frac{1}{4}$$

$y_1 \quad y_2$

18. The general solution to the homogeneous equation $t^2y'' + 7ty' + 5y = 0$ on the interval $0 < t < \infty$ is $y(t) = c_1\boxed{t^{-1}} + c_2\boxed{t^{-5}}$. A particular solution to the nonhomogeneous equation $t^2y'' + 7ty' + 5y = t$ has the form $\underbrace{y_p(t) = u_1(t)t^{-1} + u_2(t)t^{-5}}$. Which of the following are satisfied by u_1' and u_2'?

Variation of parameters

A. $u_1' = \frac{t}{4}, \quad u_2' = -\frac{t^5}{4}$

B. $u_1' = \frac{t^3}{4}, \quad u_2' = -\frac{t^7}{4}$

C. $u_1' = -t^7, \quad u_2' = t^3$

D. $u_1' = t^5, \quad u_2' = t$

E. $u_1' = -t^5, \quad u_2' = t$

$$u_1'y_1 + u_2'y_2 = 0$$

$$u_1'y_1' + u_2'y_2' = f(t)$$

↳ right side of diff. eq. in the standard form (leading coeff is 1)

$$y'' + \frac{7}{t}y' + \frac{5}{t^2}y = \frac{1}{t}$$

$y_1 = t^{-1} \quad y_2 = t^{-5}$

$$\begin{bmatrix} t^{-1} & t^{-5} & 0 \\ -t^{-2} & -5t^{-6} & \frac{1}{t} \end{bmatrix}$$

$$\xrightarrow{\frac{1}{t}R_1 + R_2} \begin{bmatrix} t^{-1} & t^{-5} & 0 \\ 0 & -4t^{-6} & t^{-1} \end{bmatrix}$$

row 2: $-4t^{-6} u_2' = t^{-1}$ $\quad u_2' = -\frac{1}{4} t^5$

row 1: $t^{-1} u_1' + t^{-5} u_2' = 0$

$$t^{-1} u_1' = -t^{-5} u_2'$$

$$u_1' = -t^{-4} u_2' = -t^{-4} \cdot -\frac{1}{4} t^5 = \frac{1}{4} t$$

---

back to $\quad u_1' y_1 + u_2' y_2 = 0$

$\quad u_1' y_1' + u_2' y_2' = \frac{1}{t}$

$$\begin{bmatrix} t^{-1} & t^{-5} \\ -t^{-2} & -5t^{-6} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{t} \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} t^{-1} & t^{-5} \\ -t^{-2} & -5t^{-6} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{t} \end{bmatrix}$$

$$= \frac{1}{\begin{vmatrix} t^{-1} & t^{-5} \\ -t^{-2} & -5t^{-6} \end{vmatrix}} \begin{bmatrix} -5t^{-6} & -t^{-5} \\ t^{-2} & t^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{t} \end{bmatrix}$$

Wronskian of t^{-1}, t^{-5}

$$= \frac{1}{-5t^{-7}+t^{-7}} \begin{bmatrix} -t^{-6} \\ t^{-2} \end{bmatrix}$$

$$= \frac{1}{-4t^{-7}} \begin{bmatrix} -t^{-6} \\ t^{-2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t \\ -\frac{1}{4}t^{5} \end{bmatrix}$$