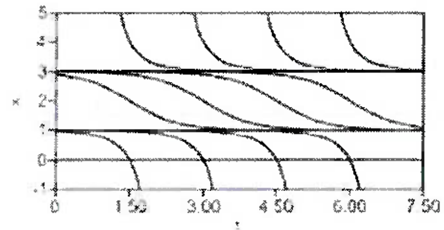
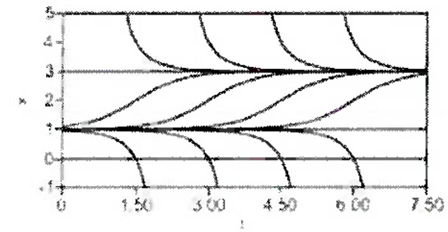
3. Select the graph which shows the solutions of the differential equation

$$x' = x^2 - 4x + 3$$

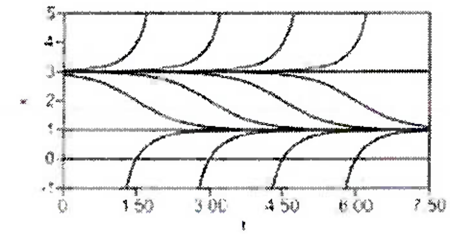
~~A.~~



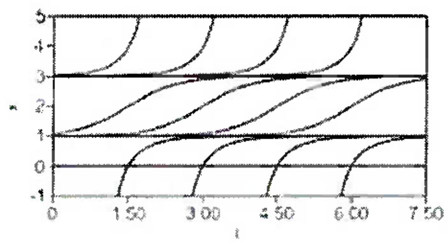
~~B.~~



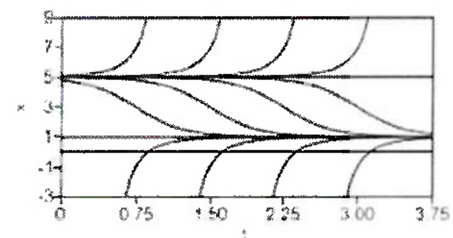
C.



~~D.~~



~~E.~~



straight lines : $x' = 0$ equilibrium solutions

$$x' = x^2 - 4x + 3 = 0$$

$(x-3)(x-1) = 0 \qquad x = 1,\ x = 3$

Stability?

$x' \quad \longrightarrow \quad 0 \quad \longleftarrow \quad 0 \quad \longrightarrow$

$x=1 \qquad x=3$

$x=0 \qquad x=2 \qquad x=5$

$x' = 0-0+3>0 \qquad x'<0 \qquad x'>0$

$x=1$ asymptotically stable

$x=3$ unstable

1. Let $y(x)$ be the solution to the initial value problem

$$\underbrace{xy' + y = 3x^2 + 2x - 4}_{\text{what kind of differential eq.?}} \qquad y(1) = 0.$$

1st-order linear

Then $y(2) =$

A. 1

B. 2

C. 3

D. 4

E. 5

$$y' + \frac{1}{x} y = 3x + 2 - \frac{4}{x}$$

integrating factor

$$I = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$\underbrace{xy' + y}_{} = 3x^2 + 2x - 4$$

$$\frac{d}{dx}(xy) = 3x^2 + 2x - 4$$

I

$$xy = \int (3x^2 + 2x - 4)\,dx$$

$$= x^3 + x^2 - 4x + C$$

$$y = x^2 + x - 4 + \frac{C}{x}$$

$$y(1) = 0 = 1 + 1 - 4 + C$$

$$= -2 + C \qquad \text{so} \qquad C = 2$$

$$y = x^2 + x - 4 + \frac{2}{x}$$

$$y(2) = 4 + 2 - 4 + 1 = 3$$

4. A container initially contains 10 L of water in which 20 g of salt is dissolved. A solution containing 4 g/L of salt is pumped into the tank at a rate of 2 L/min and the well stirred mixture runs out of the tank at a rate of 1L/min. What is the concentration of the salt in the tank after 10 minutes?

A. 1.5

B. 2.5

C. 3.5

D. 4.5

E. 5.5

let y be amount of salt in g $\quad y(0) = 20$

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= (4\ g/L)(2\ L/min) - \left(\frac{y}{10+t}\ g/L\right)(1\ L/min)$$

volume : 10 L to start
2 L/min in
1 L/min out

$$y' = 8 - \frac{y}{10+t}$$

$$y' + \frac{1}{10+t}y = 8$$

$$I = e^{\int \frac{1}{10+t}\,dt} = e^{\ln|10+t|}$$

$$= 10+t$$

$$\frac{d}{dt}\Big((10+t)(y)\Big) = 8(10+t) = 80 + 8t$$

$$(10+t)y = 80t + 4t^2 + C \qquad y(0) = 20$$

$$(10)(20) = C = 200$$

$$(10+t)y = 80t + 4t^2 + 200$$

$$y = \frac{80t + 4t^2 + 200}{10+t}$$

$$y(10) = \frac{800 + 400 + 200}{20} = \frac{1400}{20} = 70 \text{ g}$$

$$\text{concentration} = \frac{\text{amount}}{\text{volume}} = \frac{70}{\underbrace{10+10}_{\text{volume} = 10+t = 20}} = 3.5$$

19. A mass-spring system is given by the initial value problem: $y'' + 3y' + 2y = 0$, $y(0) = 2$, $y'(0) = -5$, where t is the number of seconds after the motion begins. How many seconds does it take for the mass to cross its equilibrium position for the first time?

A. $\ln 2$

B. $\ln 3$

C. $\ln 6$

D. 2

E. 3

$y'' + 3y' + 2y = 0 \quad y(0) = 2 \ , \ y'(0) = -5$

characterstic eq: $r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0$

$r = -1, \ r = -2$

Solution: $y(t) = c_1 e^{-t} + c_2 e^{-2t}$

$y'(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$

$y(0) = 2 = c_1 + c_2$

$y'(0) = -5 = -c_1 - 2c_2$

$-3 = -c_2 \quad c_2 = 3$

so, $c_1 = -1$

$y(t) = -e^{-t} + 3e^{-2t}$

underdamped : complex r
critically damped : repeated r
→ over damped : distinct

cross equilibrium: $y(t) = 0$

$$0 = -e^{-t} + 3e^{-2t}$$

$$e^{-t} = 3e^{-2t}$$

mult. by e^{t}

$$1 = 3e^{-t}$$

$$\frac{1}{3} = e^{-t}$$

$$\ln\frac{1}{3} = -t \qquad t = -\ln\frac{1}{3} = -\ln 3^{-1}$$

$$= \ln 3$$

8. Find a basis for the null space of $A = \begin{bmatrix} 1 & -2 & 2 & 3 \\ 2 & -4 & 5 & 7 \\ 3 & -6 & 3 & 6 \end{bmatrix}$.

A. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

B. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

C. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.

D. $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

E. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

null space : solution space of $A\vec{x} = \vec{0}$

$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & 3 & 0 \\ 2 & -4 & 5 & 7 & 0 \\ 3 & -6 & 3 & 6 & 0 \end{array}\right]$$

$$\to \begin{bmatrix} 1 & -2 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 \end{bmatrix}$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array}$$
$$\to \begin{bmatrix} \boxed{1} & -2 & 2 & 3 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

solution: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

x_1, x_3 have pivots NOT free

x_2, x_4 no pivots so are free

$x_2 = r \quad x_4 = t$

row 2: $x_3 = -x_4 = -t$

row 1: $x_1 = 2x_2 - 2x_3 - 3x_4$

$$x_1 = 2r - 2x_3 - 3t$$

$$= 2r + 2t - 3t = 2r - t$$

$$\vec{x} = \begin{bmatrix} 2r-t \\ r \\ -t \\ t \end{bmatrix} = r\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

what is the dimension of this null space?

$\downarrow$

# of basis vectors = 2

so it is a two-dimensional null space

12. Which of the following are vector spaces?

(i) $\{(x, y) \in \mathbb{R}^2 : \; x > y\}$

(ii) $\{(x, y) \in \mathbb{R}^2 : \; 2x - 3y = 0\}$

(iii) All solutions to $y'' + 2y = 0$ on $(-\infty, \infty)$.

(iv) All solutions to $y'' + 2 = 0$ on $(-\infty, \infty)$.

Must have :
1. existence of $\vec{0}$
2. closed under addition
3. closed under scalar multiplication

A. (i), (ii), and (iii)

B. (ii), (iii), and (iv)

C. (ii) and (iii)

D. (i) and (iv)

E. None of them are vector spaces.

i) $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $x > y$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ does not satisfy $x > y$ so no $\vec{0}$, not a vector space

also not closed under scalar mult.

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad -1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

ii) $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $2x - 3y = 0$

$3y = 2x \quad y = \frac{2}{3}x$

$\hookrightarrow \begin{bmatrix} x \\ \frac{2}{3}x \end{bmatrix}$

$\vec{0}$ included? yes

closed under addition?

one vector $\begin{bmatrix} x_1 \\ \frac{2}{3}x_1 \end{bmatrix}$

another $\begin{bmatrix} x_2 \\ \frac{2}{3}x_2 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ \frac{2}{3}x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ \frac{2}{3}x_2 \end{bmatrix} = \begin{bmatrix} (x_1+x_2) \\ \frac{2}{3}(x_1+x_2) \end{bmatrix}$$

this shows it is closed because 2nd component is still always $\frac{2}{3}$ of the 1st.

multiplication: $c\begin{bmatrix} x_1 \\ \frac{2}{3}x_1 \end{bmatrix}$

$$= \begin{bmatrix} (cx_1) \\ \frac{2}{3}(cx_1) \end{bmatrix}$$

so, closed, too.