15. Let A be a 2×2 matrix whose entries are real numbers. If $\lambda = 2 + 3i$ is a complex eigenvalue of A with corresponding complex eigenvector $\mathbf{w} = \begin{bmatrix} 1-i \\ 4 \end{bmatrix}$, then the general solution to $\mathbf{x}' = A\mathbf{x}$ is:

General solution:

$C_1 \vec{u}_1 + C_2 \vec{u}_2$, imaginary part

$\downarrow$

real part of $e^{\lambda t} \vec{w}$

A. $\mathbf{x} = C_1 e^{2t} \begin{bmatrix} \cos 3t + \sin 3t \\ 4\cos 3t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin 3t - \cos 3t \\ 4\sin 3t \end{bmatrix}$

B. $\mathbf{x} = C_1 e^{2t} \begin{bmatrix} \cos 3t - \sin 3t \\ 4\cos 3t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin 3t - \cos 3t \\ 4\sin 3t \end{bmatrix}$

C. $\mathbf{x} = C_1 e^{3t} \begin{bmatrix} \cos 2t + \sin 2t \\ 4\cos 2t \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} \sin 2t - \cos 2t \\ 4\sin 2t \end{bmatrix}$

D. $\mathbf{x} = C_1 e^{2t} \begin{bmatrix} \cos 3t + \sin 3t \\ 4\cos 3t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin 3t + \cos 3t \\ 4\sin 3t \end{bmatrix}$

E. $\mathbf{x} = C_1 e^{2t} \begin{bmatrix} \cos 3t + \sin 3t \\ 4\cos 3t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin 3t - \cos 3t \\ -4\sin 3t \end{bmatrix}$

$\lambda = 2+3i \quad \vec{w} = \begin{bmatrix} 1-i \\ 4 \end{bmatrix}$

$$\boxed{e^{it} = \cos t + i \sin t}$$

$$e^{\lambda t}\vec{w} = e^{(2+3i)t}\begin{bmatrix} 1-i \\ 4 \end{bmatrix} = e^{2t} e^{i(3t)} \begin{bmatrix} 1-i \\ 4 \end{bmatrix}$$

$$= e^{2t}(\cos 3t + i \sin 3t)\begin{bmatrix} 1-i \\ 4 \end{bmatrix}$$

$$= e^{2t}\begin{bmatrix} \cos 3t + \sin 3t + i \sin 3t - i \cos 3t \\ 4\cos 3t + i\, 4 \sin 3t \end{bmatrix}$$

$$= \underbrace{e^{2t}\begin{bmatrix} \cos 3t + \sin 3t \\ 4\cos 3t \end{bmatrix}}_{\vec{u}_1} + i \underbrace{e^{2t}\begin{bmatrix} \sin 3t - \cos 3t \\ 4 \sin 3t \end{bmatrix}}_{\vec{u}_2}$$

$$\vec{x}(t) = c_1 \vec{u}_1 + c_2 \vec{u}_2$$

13. Given that the general solution of the homogeneous equation

$$y^{(4)} + 3y^{(3)} + 3y'' + y' = 0 \text{ is } y_h(x) = C_1 + C_2e^{-x} + C_3xe^{-x} + C_4x^2e^{-x}$$

the general solution to the corresponding nonhomogeneous equation

$$y^{(4)} + 3y^{(3)} + 3y'' + y' = \underline{6x\cos x + 6xe^{-x}}$$

looks like:

A. $y(x) = y_h(x) + (Ax + B)\cos x + (Cx + D)\sin x + x^3(Ex + F)e^{-x}$

B. $y(x) = y_h(x) + (Ax + B)\cos x + (Cx + D)\sin x + x^2(Ex + F)e^{-x}$

C. $y(x) = y_h(x) + (Ax + B)\cos x + (Cx + D)\sin x + x(Ex + F)e^{-x}$

D. $y(x) = y_h(x) + x(Ax + B)\cos x + x(Dx + E)\sin x + x^4(Fx + G)e^{-x}$

E. $y(x) = y_h(x) + x(Ax + B)\cos x + x(Cx + D)\sin x + x^3(Ex + F)e^{-x}$

undetermined coeffs : 1. assume a particular solution that has the same form as the right side

2. check for any of the particular solution parts that duplicate the complementary solution (throw x at it until problem goes away)

right side : $\underbrace{6x}_{\text{linear}}\underbrace{\cos x}_{\cos x} + \underbrace{6x}_{\text{linear}}\underbrace{e^{-x}}_{\text{exponential}}$

cos x and sin x show up together

$$y_p = (Ax+B)\cos x + (Cx+D)\sin x + (Ex+F)e^{-x}$$

$$y_c = c_1 + c_2 e^{-x} + c_3 x e^{-x} + c_4 x^2 e^{-x}$$

$(Ex+F)e^{-x}$ is duplicating parts of y_c

multiply by x: $x(Ex+F)e^{-x}$ not enough

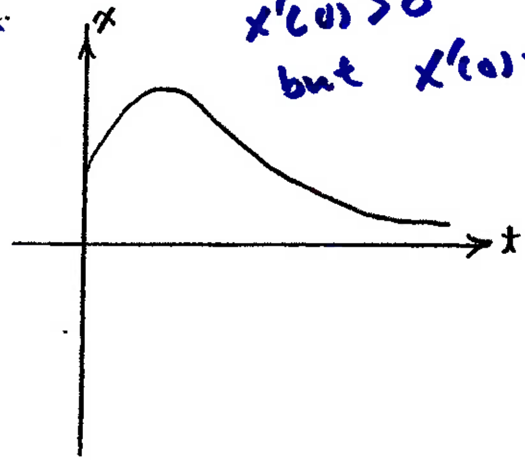
one more: $x^2(Ex+F)e^{-x}$ not enough, Fx^2e^{-x} is still duplicating

one more: $x^3(Ex+F)e^{-x}$ good now

$$y_p = (Ax+B)\cos x + (Cx+D)\sin x + x^3(Ex+F)e^{-x}$$

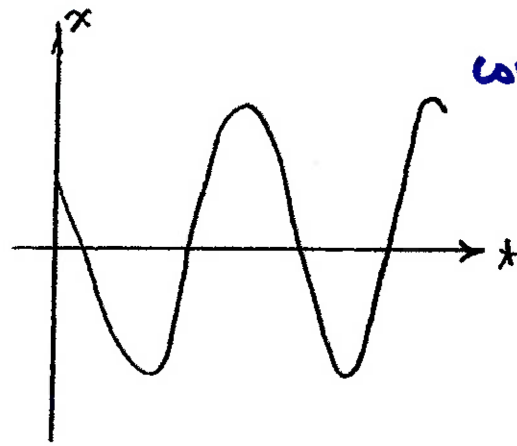
21. The oscillation of a spring-mass system is determined by $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$, with initial conditions $x(0) = 1$ and $\frac{dx}{dt}(0) = -3$. Then a sketch of the motion $x(t)$ is

~~A.~~



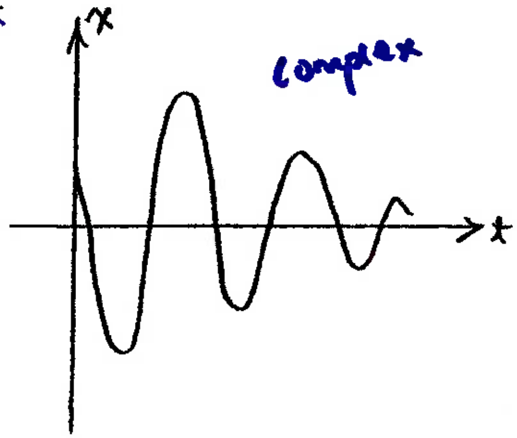
$x'(0) > 0$ but $x'(0) = -3$

~~B.~~



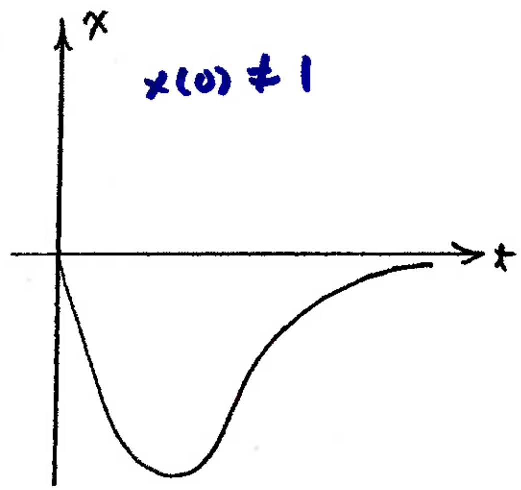
complex

~~C.~~



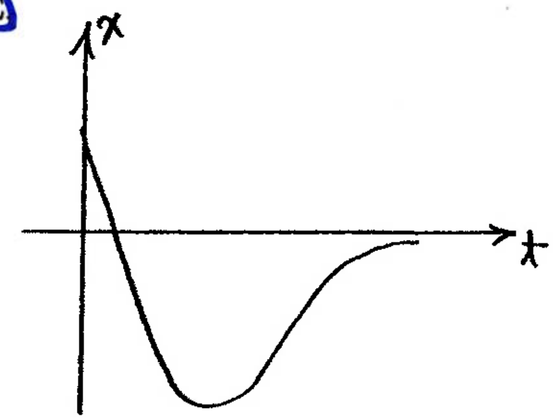
complex

~~D.~~



$x(0) \neq 1$

E.



$x'' + 3x' + 2x = 0 \qquad x(0) = 1 \qquad x'(0) = -3$

$r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0 \qquad r = -1,\ r = -2$

$x(t) = C_1 e^{-t} + C_2 e^{-2t}$

$x'(t) = -C_1 e^{-t} - 2C_2 e^{-2t}$

$x(0) = 1 = C_1 + C_2 \qquad -2 = -C_2 \qquad C_2 = 2$

$x'(0) = -3 = -C_1 - 2C_2 \qquad$ so $C_1 = -1$

$x(t) = -e^{-t} + 2e^{-2t}$

underdamp → complex only one w/ oscillation

critically damp → repeated

overdamp → distinct

4. The solution of $\underbrace{(3x^2+y)dx+(x+2y)dy=0}$ passing through the point $(1,1)$ is

A. $x^2+xy+y^2=3$

B. $x^2+xy+y^3=3$

C. $x^2+x+y^2=3$

D. $x^3+xy+y^2=3$

E. $x^3+x^2y+y^3=3$

possibly exact : $M\,dx+N\,dy=0$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

is this exact ?

$M=3x^2+y \qquad N=x+2y$

$M_y=1 \qquad N_x=1 \qquad$ so is exact.

exact : $f(x,y)=C$ such that $M=f_x \quad N=f_y$

$f_x=M=3x^2+y \longrightarrow \int(3x^2+y)\,dx = x^3+xy+g(y)$, take partial w/ y compare to N

$f_y= \quad x+2y \quad = \cancel{3x^2} + x + \frac{dg}{dy}$

so, $\frac{dg}{dy}=2y \to g=y^2$

solution: $x^3+xy+y^2=C$

through $(1,1)$: $1+1+1=C=3$

$\boxed{x^3+xy+y^2=3}$

15. The matrix $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ has repeated eigenvalues of $\lambda = 1$ and $\lambda = 1$. If $\mathbf{x}(t)$ is the solution to system $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{x}$ with initial condition $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, find $\mathbf{x}(1)$

A. $\begin{bmatrix} 2 \\ 2e \end{bmatrix}$

B. $\begin{bmatrix} e \\ 2e \end{bmatrix}$

C. $\begin{bmatrix} -e \\ 2e \end{bmatrix}$

D. $\begin{bmatrix} 2e \\ e \end{bmatrix}$

E. $\begin{bmatrix} 2e \\ -e \end{bmatrix}$

$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad \lambda = 1, \ \lambda = 1$

$(A - \lambda I)\vec{v} = \vec{0}$ $\quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

one free variable
→ one eigenvector

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

find $\vec{v}_2$: $\quad (A - \lambda I)\vec{v}_2 = \vec{v}_1$

$(A - \lambda I)^2 \vec{v}_2 = \vec{0}$

$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right]$ $\quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\vec{x}(t) = c_1 e^{\lambda t}\vec{v}_1 + c_2 e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)$$

$$= c_1 e^{t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{t}\left(t\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}\right)$$

$$\vec{x}(0) = \begin{bmatrix}1\\0\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\0\end{bmatrix} \qquad \begin{aligned} c_1 &= 0 \\ c_2 &= 1 \end{aligned}$$

$$\vec{x}(t) = e^{t}\left(t\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}\right)$$

$$\vec{x}(1) = e\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2e\\e\end{bmatrix}$$

11. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ and let $B = A^{-1}$. Then the entry b_{13} of A^{-1} is

col 3, Row 1

A. 0

B. 1

C. -1

D. 2

E. -2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \cdots \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right]$$

Or:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$b_{13} + b_{33} = 0$ $\qquad b_{13} = 1$

$b_{23} = 0$

$b_{23} - b_{33} = 1 \rightarrow b_{33} = -1$

$$y' + \frac{3}{x} y = x^2 y^{2}$$ n

Bernoulli: $y' + p(x) y = g(x) y^n$

$n \neq 0,\ n \neq 1$

subs: $u = y^{1-n} = y^{1-2} = y^{-1} = \frac{1}{y}$

rewrite in terms of u

$u = \frac{1}{y}$ $\quad u' = -y^{-2} y'$ $\quad y' = -y^2 u'$

$$-y^2 u' + \frac{3}{x} y = x^2 y^2$$

$$u' - \frac{3}{x} y^{-1} = -x^2$$ u

$u' - \frac{3}{x} u = -x^2$ linear in u

integrating factor: $I = e^{\int -\frac{3}{x} dx} = e^{-3\ln x}$

$$= e^{\ln x^{-3}} = x^{-3}$$

$$\frac{d}{dx}\left(x^{-3}u\right) = -x^{-1}$$

$$x^{-3}u = -\ln x + C$$

$$u = -x^3 \ln x + Cx^3 = \frac{1}{y} \quad \boxed{y = \frac{1}{-x^3\ln x + Cx^3}}$$

$y'' + (\tan x) y' = \cos^2 x$ $\qquad$ $y'' + (\tan x) y' = 0$

$y_1 = \sin x \qquad y_2 = 1$

particular: $y_p = u_1 \cdot \sin x + u_2 \cdot 1 = u_1 y_1 + u_2 y_2$

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = f(x)$$

right side of diff. eq. w/ leading coeff = 1

$$\begin{bmatrix} \sin x & 1 \\ \cos x & 0 \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \cos^2 x \end{bmatrix}$$

$$\begin{bmatrix} \sin x & 1 & 0 \\ \cos x & 0 & \cos^2 x \end{bmatrix} \xrightarrow{-\frac{\cos x}{\sin x} R_1 + R_2} \begin{bmatrix} \sin x & 1 & 0 \\ 0 & -\frac{\cos x}{\sin x} & \cos^2 x \end{bmatrix}$$

$$-\frac{\cos x}{\sin x} u_2' = \cos^2 x$$

$$u_2' = -\sin x \cos x$$

$$u_1 = \sin x$$

$$\sin x \, u_1' + u_2' = 0$$

$$u_1' = -\frac{u_2'}{\sin x} = \cos x$$