12. Which of the following are vector spaces?

(i) $\{(x, y) \in \mathbb{R}^2 : \; x > y\}$

(ii) $\{(x, y) \in \mathbb{R}^2 : \; 2x - 3y = 0\}$

(iii) All solutions to $y'' + 2y = 0$ on $(-\infty, \infty)$.

(iv) All solutions to $y'' + 2 = 0$ on $(-\infty, \infty)$.

Must have :

1. existence of $\vec{0}$
2. closed under addition
3. closed under scalar multiplication

A. (i), (ii), and (iii)

B. (ii), (iii), and (iv)

C. (ii) and (iii)

D. (i) and (iv)

E. None of them are vector spaces.

i) $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $x > y$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ does not satisfy $x > y$ so no $\vec{0}$, not a vector space

also not closed under scalar mult.

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad -1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

ii) $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $2x - 3y = 0$

$3y = 2x \quad y = \frac{2}{3}x$

$\hookrightarrow \begin{bmatrix} x \\ \frac{2}{3}x \end{bmatrix}$

$\vec{0}$ included? yes

closed under addition?

one vector $\begin{bmatrix} x_1 \\ \frac{2}{3}x_1 \end{bmatrix}$

another $\begin{bmatrix} x_2 \\ \frac{2}{3}x_2 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ \frac{2}{3}x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ \frac{2}{3}x_2 \end{bmatrix} = \begin{bmatrix} (x_1+x_2) \\ \frac{2}{3}(x_1+x_2) \end{bmatrix}$$

this shows it is closed because 2nd component is still always $\frac{2}{3}$ of the 1st.

multiplication: $c\begin{bmatrix} x_1 \\ \frac{2}{3}x_1 \end{bmatrix}$

$$= \begin{bmatrix} (cx_1) \\ \frac{2}{3}(cx_1) \end{bmatrix}$$ so, closed, too.

iii) solutions to $\underbrace{y''+2y=0}$ on $(-\infty, \infty)$

char. eq. $r^2+2=0 \quad r=\pm\sqrt{2}\,i$

$y = \underbrace{c_1\cos(\sqrt{2}\,t) + c_2\sin(\sqrt{2}\,t)}$

solution space spanned by the "vectors" $\cos(\sqrt{2}\,t)$ and $\sin(\sqrt{2}\,t)$

does $\vec{0}$ exist? yes, for example $t=0 \rightarrow \sin(\sqrt{2}t)=0$

addition: is the sum of $\sin(\sqrt{2}t)$ and $\cos(\sqrt{2}t)$ still

a solution of $y''+2y=0$? yes, general solution

is a linear combo of $\sin(\sqrt{2}t)$ and $\cos(\sqrt{2}t)$

mult.: if $\sin(\sqrt{2}t)$ is a solution, so is $c_1 \sin(\sqrt{2}t)$

(gen. solution is linear combo)

iv) Solutions to $y''+2=0$

is $y=0$ in the solution space?

no, since $y=0$ is not a solution to $y''+2=0$

there is no $\vec{0}$, so not a vector space

18. The general solution to the homogeneous equation $t^2y'' + 7ty' + 5y = 0$ on the interval $0 < t < \infty$ is $y(t) = c_1t^{-1} + c_2t^{-5}$. A particular solution to the nonhomogeneous equation $t^2y'' + 7ty' + 5y = t$ has the form $y_p(t) = u_1(t)t^{-1} + u_2(t)t^{-5}$. Which of the following are satisfied by u_1' and u_2'?

A. $u_1' = \frac{t}{4}$, $\quad u_2' = -\frac{t^5}{4}$

B. $u_1' = \frac{t^3}{4}$, $\quad u_2' = -\frac{t^7}{4}$

C. $u_1' = -t^7$, $\quad u_2' = t^3$

D. $u_1' = t^5$, $\quad u_2' = t$

E. $u_1' = -t^5$, $\quad u_2' = t$

variation of parameters

solution is $y = u_1y_1 + u_2y_2$

here, $y_1 = t^{-1}$ $\quad y_2 = t^{-5}$

System of eqs. to find u_1, u_2:

$$u_1'y_1 + u_2'y_2 = 0$$

$$u_1'y_1' + u_2'y_2' = f(t)$$

right side of the differential eq. in the standard form (leading coeff is 1)

$$t^2y'' + 7ty' + 5y = t$$

$$y'' + \frac{7}{t}y' + \frac{5}{t^2}y = \frac{1}{t} \rightarrow f(t)$$

system: $u_1'y_1 + u_2'y_2 = 0$ $\qquad y_1 = t^{-1} \qquad y_2 = t^{-5}$

$u_1'y_1' + u_2'y_2' = \frac{1}{t}$

$$\begin{bmatrix} t^{-1} & t^{-5} \\ -t^{-2} & -5t^{-6} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{t} \end{bmatrix}$$

option 1: row reduction

$$\begin{bmatrix} t^{-1} & t^{-5} & 0 \\ -t^{-2} & -5t^{-6} & \frac{1}{t} \end{bmatrix} \xrightarrow{t^{-1}R_1 + R_2} \begin{bmatrix} t^{-1} & t^{-5} & 0 \\ 0 & -4t^{-6} & t^{-1} \end{bmatrix}$$

row 2: $-4t^{-6}u_2' = t^{-1}$

$$u_2' = -\frac{1}{4}t^5$$

row 1: $t^{-1}u_1' + t^{-5}u_2' = 0$

$$t^{-1}u_1' = -t^{-5} \cdot -\tfrac{1}{4}t^{5} = \tfrac{1}{4}$$

$$u_1' = \tfrac{1}{4}t$$

option 2: use matrix inverse

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} t^{-1} & t^{-5} \\ -t^{-2} & -5t^{-6} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ t^{-1} \end{bmatrix}$$

$$= \frac{1}{\begin{vmatrix} t^{-1} & t^{-5} \\ -t^{-2} & -5t^{-6} \end{vmatrix}} \begin{bmatrix} -5t^{-6} & -t^{-5} \\ t^{-2} & t^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ t^{-1} \end{bmatrix}$$

Wronskian of
$y_1 = t^{-1}$, $y_2 = t^{-5}$

$$= \frac{1}{-5t^{-7} + t^{-7}} \begin{bmatrix} -t^{-6} \\ t^{-2} \end{bmatrix} = \begin{bmatrix} \tfrac{1}{4}t \\ -\tfrac{1}{4}t^{5} \end{bmatrix}$$

13. Given that the general solution of the homogeneous equation

$$y^{(4)} + 3y^{(3)} + 3y'' + y' = 0 \text{ is } y_h(x) = C_1 + C_2e^{-x} + C_3xe^{-x} + C_4x^2e^{-x}$$

the general solution to the corresponding nonhomogeneous equation

$$y^{(4)} + 3y^{(3)} + 3y'' + y' = \underline{6x\cos x + 6xe^{-x}}$$

looks like:

A. $y(x) = y_h(x) + (Ax + B)\cos x + (Cx + D)\sin x + x^3(Ex + F)e^{-x}$

B. $y(x) = y_h(x) + (Ax + B)\cos x + (Cx + D)\sin x + x^2(Ex + F)e^{-x}$

C. $y(x) = y_h(x) + (Ax + B)\cos x + (Cx + D)\sin x + x(Ex + F)e^{-x}$

D. $y(x) = y_h(x) + x(Ax + B)\cos x + x(Dx + E)\sin x + x^4(Fx + G)e^{-x}$

E. $y(x) = y_h(x) + x(Ax + B)\cos x + x(Cx + D)\sin x + x^3(Ex + F)e^{-x}$

undetermined coefficients

first, look at right side of diff. eq and come up with a suitable form of particular solution

$$\underbrace{6x\cos x}_{\text{linear}} + \underbrace{6xe^{-x}}$$

need $\sin x$, too

$$y_p = \overbrace{(Ax+B)\cos x + (Cx+D)\sin x} + (Ex+F)e^{-x}$$

then check if any of that is duplicating the complementary solution $c_1 + c_2e^{-x} + c_3xe^{-x} + c_4x^2e^{-x}$

$(Ex+F)e^{-x}$ duplicates comp. solution

to fix: multiply by x (possibly multiple times)

here, we need to do it $\underline{3}$ times

$$y_p = (Ax+B)\cos x + (Cx+D)\sin x + x^3(Ex+F)e^{-x}$$

15. Let A be a 2×2 matrix whose entries are real numbers. If $\lambda = 2 + 3i$ is a complex eigenvalue of A with corresponding complex eigenvector $\mathbf{w} = \begin{bmatrix} 1-i \\ 4 \end{bmatrix}$, then the general solution to $\mathbf{x}' = A\mathbf{x}$ is:

A. $\mathbf{x} = C_1 e^{2t} \begin{bmatrix} \cos 3t + \sin 3t \\ 4\cos 3t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin 3t - \cos 3t \\ 4\sin 3t \end{bmatrix}$

B. $\mathbf{x} = C_1 e^{2t} \begin{bmatrix} \cos 3t - \sin 3t \\ 4\cos 3t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin 3t - \cos 3t \\ 4\sin 3t \end{bmatrix}$

C. $\mathbf{x} = C_1 e^{3t} \begin{bmatrix} \cos 2t + \sin 2t \\ 4\cos 2t \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} \sin 2t - \cos 2t \\ 4\sin 2t \end{bmatrix}$

D. $\mathbf{x} = C_1 e^{2t} \begin{bmatrix} \cos 3t + \sin 3t \\ 4\cos 3t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin 3t + \cos 3t \\ 4\sin 3t \end{bmatrix}$

E. $\mathbf{x} = C_1 e^{2t} \begin{bmatrix} \cos 3t + \sin 3t \\ 4\cos 3t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin 3t - \cos 3t \\ -4\sin 3t \end{bmatrix}$

complex λ: form one solution $e^{\lambda t}\vec{w}$, isolate the real and imaginary parts

then general solution is a linear combo of those parts.

$e^{it} = \cos(t) + i\sin(t)$

$$e^{(2+3i)t}\begin{bmatrix} 1-i \\ 4 \end{bmatrix} = e^{2t}e^{i(3t)}\begin{bmatrix} 1-i \\ 4 \end{bmatrix}$$

$$= e^{2t}(\cos 3t + i\sin 3t)\begin{bmatrix} 1-i \\ 4 \end{bmatrix}$$

$$= e^{2t}\left(\begin{bmatrix} \cos 3t + \sin 3t + i\sin 3t - i\cos 3t \\ 4\cos 3t + i\,4\sin 3t \end{bmatrix}\right)$$

$$= \boxed{e^{2t}\begin{bmatrix} \cos 3t + \sin 3t \\ 4\cos 3t \end{bmatrix}} + i\,\boxed{e^{2t}\begin{bmatrix} \sin 3t - \cos 3t \\ 4\sin 3t \end{bmatrix}}$$

gen. sol is linear combo of

19. Find all constants b such that the origin is a spiral source of the system

$$X'(t) = \begin{bmatrix} 3 & b \\ 1 & 4 \end{bmatrix} X(t), \quad b \text{ in } \mathbb{R}$$

are

A. $b < -\frac{1}{3}$

B. $b > -\frac{1}{3}$

C. $-\frac{1}{4} < b < -\frac{1}{4}$

D. $b > -\frac{1}{4}$

E. $b < -\frac{1}{4}$

Spiral source : complex and positive real part

(sink if negative real part)

$$\begin{vmatrix} 3-\lambda & b \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(4-\lambda) - b = 0$$

$$\lambda^2 - 7\lambda + (12-b) = 0$$

$$\lambda = \frac{7 \pm \sqrt{49-4(12-b)}}{2}$$

$49-4(12-b) < 0$ (complex λ)

$49 < 4(12-b)$

$49 < 48-4b$

$4b < -1 \qquad b < -\frac{1}{4}$

note real part of λ is always $\frac{7}{2}$

so ~~b~~ changing b cannot make this a sink