4. The solution of $(3x^2+y)dx+(x+2y)dy=0$ passing through the point $(1,1)$ is

A. $x^2+xy+y^2=3$

B. $x^2+xy+y^3=3$

C. $x^2+x+y^2=3$

D. $x^3+xy+y^2=3$

E. $x^3+x^2y+y^3=3$

looks exact

$$\boxed{\begin{array}{l} M\,dx + N\,dy = 0 \\ \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \end{array}}$$

$M = 3x^2+y \qquad N = x+2y$

$M_y = 1 \qquad N_x = 1$ so this eq. is exact.

exact: $f(x,y)=C$ such that $\dfrac{\partial f}{\partial x} = M \quad \dfrac{\partial f}{\partial y} = N$

$$\frac{\partial f}{\partial x} = M = 3x^2+y$$

$$\frac{\partial f}{\partial y} = N = x+2y$$

$$f = \int M\,dx = \int (3x^2 + y)\,dx = x^3 + xy + g(y)$$

$$\frac{\partial f}{\partial y} = x + \frac{dg}{dy} = x + 2y \rightarrow \frac{dg}{dy} = 2y \quad \text{so} \quad g(y) = y^2$$

so, solution is $f(x,y) = x^3 + xy + y^2 = C$

through $(1,1)$: $\quad 1 + 1 + 1 = C = 3$

$$x^3 + xy + y^2 = 3$$

15. The matrix $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ has repeated eigenvalues of $\lambda = 1$ and $\lambda = 1$. If $\mathbf{x}(t)$ is the solution to system $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{x}$ with initial condition $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, find $\mathbf{x}(1)$

A. $\begin{bmatrix} 2 \\ 2e \end{bmatrix}$

B. $\begin{bmatrix} e \\ 2e \end{bmatrix}$

C. $\begin{bmatrix} -e \\ 2e \end{bmatrix}$

D. $\begin{bmatrix} 2e \\ e \end{bmatrix}$

E. $\begin{bmatrix} 2e \\ -e \end{bmatrix}$

enough eigenvectors ?

$(A-\lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

one free variable
→ one eigenvector
(missing one)

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

generalized eigenvector: $(A-\lambda I)\vec{v}_2 = \vec{v}_1$

$(A-\lambda I)^2\vec{v}_2 = \vec{0}$

$$(A-\lambda I)\vec{v}_2 = \vec{v}_1$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1+r \\ r \end{bmatrix} = \underset{(r=0)}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

solution: $\vec{x}(t) = c_1 e^{\lambda t}\vec{v}_1 + c_2 e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)$

$$= c_1 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{t}\left(t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{matrix} c_1 = 0 \\ c_2 = 1 \end{matrix}$$

$$\vec{x}(t) = e^{t}\left(t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$\vec{x}(1) = e\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2e \\ e \end{bmatrix}$$

3x3 missing two: $\vec{v}_1$ = ordinary eigenvector

$$(A-\lambda I)\vec{v}_2 = \vec{v}_1$$

$$(A-\lambda I)\vec{v}_3 = \vec{v}_2$$

$$(A-\lambda I)^3 \vec{v}_3 = \vec{0}$$

$\vec{v}_2, \vec{v}_3$: generalized eigenvectors

Solution: $\vec{x}(t) = c_1 e^{\lambda t}\vec{v}_1 + c_2 e^{\lambda t}(t\vec{v}_1 + \vec{v}_2) + c_3 e^{\lambda t}\left(\frac{t^2}{2}\vec{v}_1 + t\vec{v}_2 + \vec{v}_3\right)$

3x3 missing one:

$$\vec{x}' = \begin{bmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{bmatrix} \vec{x}$$

$$\lambda = 1, 1, 1$$

$$\vec{v}_0 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

missing: $\vec{v}_3$ such that $(A-\lambda I)\vec{v}_3 = \vec{u}$

$\downarrow$

linear combo of ordinary eigenvectors

Solution: $\vec{x}(t) = c_1 e^{\lambda t}\vec{v}_1 + c_2 e^{\lambda t}\vec{v}_2 + c_3 e^{\lambda t}(t\vec{u} + \vec{v}_3)$

ordinary eigenvectors

$$(A-\lambda I)\vec{v}_3 = \vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2$$

$$\underbrace{(A-\lambda I)^2}_{0 \text{ matrix}}\vec{v}_3 = c_1 \underbrace{(A-\lambda I)\vec{v}_1}_{\vec{0}} + c_2 \underbrace{(A-\lambda I)\vec{v}_2}_{\vec{0}} = \vec{0}$$

choose $\vec{v}_3$ so it is linearly independent from $\vec{v}_1, \vec{v}_2$

choose $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

then $(A-\lambda I)\vec{v}_3 = \vec{u} = \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}$

solution: $\vec{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} + c_3 e^t \left(t \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$

$$\vec{x}' = A\vec{x}$$

$$\vdots$$

$$\vec{x}(t) = \cdots = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \end{bmatrix}$$

F19 #20

$F = kx$

$2 = k \cdot \frac{1}{2}$ $\quad k = 4$

no damping: $\quad mx'' + kx = 0$

weight = 2 lb

mass? $\quad 2 = m \cdot 32 \quad m = \frac{1}{16}$

$\frac{1}{16}x'' + 4x = 0$

$x'' + 64x = 0 \quad x(0) = \frac{1}{4} \quad x'(0) = 0$

$r^2 + 64 = 0 \quad r = \pm 8i$

$x(t) = C_1 \cos(8t) + C_2 \sin(8t)$

$x'(t) = -8C_1 \sin(8t) + 8C_2 \cos(8t)$

$\frac{1}{4} = C_1$

$0 = C_2 \quad x(t) = \frac{1}{4}\cos(8t)$