3.4 Matrix Operations

notation: usually capital letters A, B, etc

in print, usually boldfaced

each number is called an element

each element of A is usually denoted by its lowercase

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = [a_{ij}]$$

i^{th} row

j^{th} column

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$a_{11} = 1$, $a_{12} = 2$, etc

a lot of matrix operations are like operations w/ numbers (scalars)

<u>addition</u> : if two matrices are <u>of the same size</u>

addition is element by element

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} -3 & -2 \\ 5 & 10 \end{bmatrix}$ $C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

2×2 2×2 3×1

$$A + B = \begin{bmatrix} 1+-3 & 2+-2 \\ 3+5 & 4+10 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 8 & 14 \end{bmatrix}$$

but $A+C$ is not possible, neither is $B+C$ because the sizes do not match.

<u>scalar multiplication</u> : constant c, matrix A

then $cA = [c\,a_{ij}]$ matrix of the same size as A each element multiplied by c

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $5A = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}$

$B = \begin{bmatrix} -3 & -2 \\ 5 & 10 \end{bmatrix}$ $-B = (-1)B = \begin{bmatrix} 3 & 2 \\ -5 & -10 \end{bmatrix}$

$C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $10C = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$

Subtraction: $A - B = A + (-1)B$

must have the same dimension

$$A + (-1)B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + (-1)\begin{bmatrix} -3 & -2 \\ 5 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ -2 & -6 \end{bmatrix}$$

a matrix that is just one row or one column is called a vector

$C = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ is a row vector

$D = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is a column vector matrix multiplication : AB is defined only the number of columns of A = the number of rows of B

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

2×3 rows cols $\qquad$ 3×1 rows col

since # cols of A = # rows of B, AB is possible (but BA is NOT) → order of multiplication matters!

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

multiply row of A by column of B term by term and add

$2 \times \boxed{3}$ $\quad$ $\boxed{3} \times 1$

inner dimensions MUST match

$$= \begin{bmatrix} 1\cdot 1 + 2\cdot 4 + 3\cdot 5 \\ 0\cdot 1 + -1\cdot 4 + -2\cdot 5 \end{bmatrix} = \begin{bmatrix} 24 \\ -14 \end{bmatrix} = AB$$

2×1

<u>example</u> $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -4 & 0 & 3 \\ 1 & -5 & 2 \end{bmatrix}$

2×2 $\quad$ 2×3

is AB possible? yes (result is 2×3)

is BA possible? no

$$AB = \begin{bmatrix} \boxed{1 \quad 1} \\ 2 \quad 1 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} -4 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 0 \\ -5 \end{matrix}} & \boxed{\begin{matrix} 3 \\ 2 \end{matrix}} \end{bmatrix}$$

$$= \begin{bmatrix} -4+1 & 0-5 & 3+2 \\ -7 & -5 & 8 \end{bmatrix} = \begin{bmatrix} -3 & -5 & 5 \\ -7 & -5 & 8 \end{bmatrix}$$

$AB \neq BA$ in general, even if BOTH are possible

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ 2×2 $\quad$ $AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 9 \\ -5 & -5 \end{bmatrix} = \begin{bmatrix} -7 & 13 \\ -6 & 4 \end{bmatrix}$

$B = \begin{bmatrix} -1 & 9 \\ -5 & -5 \end{bmatrix}$ 2×2 $\quad$ $BA = \begin{bmatrix} -1 & 9 \\ -5 & -5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ -15 & -10 \end{bmatrix}$

of course, AB can equal BA but in general $AB \neq BA$

$$\boxed{A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

"identity matrix" → equivalent of the number 1 $\quad 1 \cdot a = a$

$a \cdot 1 = a$

I (square matrix of any size that makes operation possible)

$$\underset{5\times5}{I}\ \underset{5\times5}{B} = \underset{5\times5}{B}\ \underset{5\times5}{I}$$

there is NO matrix division → equivalent is matrix inverse (later part of chapter)

ways to represent a linear system

$$\left.\begin{aligned} 2x_1 + 3x_2 - 4x_3 &= 1 \\ -x_2 + 5x_3 &= 2 \\ 3x_1 + 4x_2 &= 3 \end{aligned}\right\}$$

intersection of 3 planes
(one interpretation)

as a matrix equation :

$$\begin{bmatrix} 2 & 3 & -4 \\ 0 & -1 & 5 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

multiplied out :

$$\begin{bmatrix} 2x_1 + 3x_2 - 4x_3 \\ -x_2 + 5x_3 \\ 3x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

notice it is also

$$x_1\begin{bmatrix}2\\0\\3\end{bmatrix}+x_2\begin{bmatrix}3\\-1\\4\end{bmatrix}+x_3\begin{bmatrix}-4\\5\\0\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

this form is called <u>vector equation</u>

when vectors are added w/ constant multiples, we call ~~this~~ a <u>linear combination</u>

second interpretation: find x_1, x_2, x_3 such that

$$x_1\begin{bmatrix}2\\0\\3\end{bmatrix}+x_2\begin{bmatrix}3\\-1\\4\end{bmatrix}+x_3\begin{bmatrix}-4\\5\\0\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$