3.5 Inverses of Matrices

equivalent to scalar inverse : $a \cdot a^{-1} = a^{-1} \cdot a = 1 \quad a \neq 0$

for matrix, $AA^{-1} = I = A^{-1}A$

↳ identity matrix → $n \times n$ matrix with ones on it <u>main diagonal</u> and zeros everywhere else

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad 2\times 2 \text{ identity matrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 3\times 3 \text{ identity}$$

size is such that $\underset{n\times n}{A}\,\underset{n\times n}{A^{-1}} = \underset{n\times n}{I}$ so based on A

So, for A to have an inverse ("A is invertible") it must be square

not every square matrix has an inverse (if it doesn't, A is "singular") for 2×2, it's inverse has a simple formula

$$\boxed{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

$ad-bc$ is the <u>determinant</u> of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

notice $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible only if it determinant is <u>NOT</u> zero

<u>example</u>

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A^{-1} = \frac{1}{1\cdot 4 - 2\cdot 3}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

check: $AA^{-1} = I \qquad A^{-1}A = I$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

scalar : $ax = b \rightarrow x = a^{-1}b \quad (a \neq 0) \qquad 3x = 2 \rightarrow x = 3^{-1}2 = \frac{2}{3}$

matrix : $A\vec{x} = \vec{b} \rightarrow \vec{x} = A^{-1}\vec{b}$ (or we can do Gaussian elimination)

<u>example</u>

$$x_1 + 2x_2 = 5$$
$$3x_1 + 4x_2 = 6$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 5 \\ 6 \end{bmatrix}}_{\vec{b}}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\vec{b} = \frac{1}{1\cdot 4 - 2\cdot 3}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 9/2 \end{bmatrix}$$

no simple formulas for A^{-1} if $n > 2$ (A is $n \times n$)

instead, there is an algorithm (which works for all $n \times n$ including 2×2)

$n \times n$ A, make an augmented matrix this way

$$\left[A \;\vdots\; I \right]$$

I is also $n \times n$

(this augmented matrix is $n \times 2n$)

do row operations until the left half is I, whatever on the right is A^{-1}

$$\left[A \;\vdots\; I \right] \to \cdots \to \left[I \;\vdots\; A^{-1} \right]$$

try it on $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(-3)R_1+R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{bmatrix}$$

$$\xrightarrow{(-\frac{1}{2})R_2} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right] \quad A^{-1}$$

<u>example</u>

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad \text{find } A^{-1} \text{ if it exists}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{(-1)R_1+R_3}{\xrightarrow{(-1)R_1+R_2}} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(-1)R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad A^{-1}$$

$$I$$

example

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -10 & -5 & 1 \end{bmatrix}$$

no pivot
so left half
will never become I

$$\longrightarrow \left[\begin{array}{c:c} I & A^{-1} \end{array}\right]$$

since left side is never I
we cannot get here $\rightarrow$ A^{-1} does <u>not</u> exist
(A is not invertible or A is singular)

so, $A\vec{x} = \vec{b}$ we cannot do $\vec{x} = A^{-1}\vec{b}$ (either no solution or infinitely-many solutions)

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why does $[A \;\; I] \rightarrow$ elimination $\rightarrow [I \;\; A^{-1}]$ work?

revisit a simpler system:

$$a_1 x_1 + b_1 x_2 = c_1$$
$$a_2 x_1 + b_2 x_2 = c_2$$

$$\left[\begin{array}{cc:c} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{c:c} I & \begin{matrix} x_1 \\ x_2 \end{matrix} \end{array}\right]$$

finding the inverse of A is very similar, but w/ more columns

$$X = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}$$ goal: find c_1, c_2, d_1, d_2

$$AX = I \qquad (X = A^{-1})$$

$$\left[\begin{array}{cc:c} a_1 & b_1 & \cancel{c_1}\,1 \\ a_2 & b_2 & \cancel{c_2}\,0 \end{array}\right] \rightarrow \left[\begin{array}{c:c} I & \text{1st col of } A^{-1} \end{array}\right]$$

1st col of I

$$\text{1st col of } A^{-1} = \begin{matrix} c_1 \\ c_2 \end{matrix}$$

$$\left[\begin{array}{cc:cc} a_1 & b_1 & \cancel{d_1} & 0 \\ a_2 & b_2 & \cancel{d_2} & 1 \end{array}\right] \rightarrow \left[\begin{array}{c:c} I & \text{2nd col of } A^{-1} \end{array}\right]$$

2nd col of I

$$\begin{array}{c} \text{"} \\ d_1 \\ d_2 \end{array}$$

can combine into

$$\left[\begin{array}{cc:cccc} a_1 & b_1 & 1 & \cancel{c_1} & \cancel{d_1} & 0 \\ a_2 & b_2 & 0 & \cancel{c_2} & \cancel{d_2} & 1 \end{array}\right]$$

columns do not interact

$$\rightarrow \cdots \rightarrow \left[\begin{array}{c:cc} I & \text{1st col} & \text{2nd col} \end{array}\right]$$

$$\underbrace{\text{1st col} \quad \text{2nd col}}_{A^{-1} = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}}$$