3.6 Determinants (part 1)

Scalar: a^{-1} exists if $a \neq 0$ $\quad a^{-1} = \frac{1}{a}$ $(a \neq 0)$

matrix: A^{-1} exists if <u>determinant</u> of A ~~exists~~ is not zero

if A is 2x2, its determinant, $\det A = \det(A) = |A|$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$A^{-1} = \frac{1}{\underbrace{ad-bc}_{\det A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

for 3x3 and beyond, no formula like $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$ exists

however, using <u>cofactor expansion</u> we can find the determinant of 3x3 as sum of several 2x2

(4x4 as sum of several 3x3 which are sum of 2x2)

Cofactor expansion

example $A = \begin{bmatrix} 2 & 0 & 4 \\ 3 & 4 & 2 \\ 0 & 4 & -2 \end{bmatrix}$ find det A

choose ANY one row/column to expand

here, let's use column 1

row 1 col 1 a_{21} a_{31}

$$\det A = (2)(-1)^{1+1}\underbrace{\begin{vmatrix} 4 & 2 \\ 4 & -2 \end{vmatrix}}_{\text{det of sub matrix after blocking out row 1, col 1}} + (3)(-1)^{2+1}\begin{vmatrix} 0 & 4 \\ 4 & -2 \end{vmatrix} + (0)(-1)^{3+1}\begin{vmatrix} 0 & 4 \\ 4 & 2 \end{vmatrix}$$

1st # in col 1

$a_{11} = 2$

$$= (2)(1)(-8-8) + (3)(-1)(-16) + (0)$$

$$= \boxed{16}$$

$(-1)^{i+j}$ distributes signs like this $\begin{bmatrix} 2^{+} & 0^{-} & 4^{+} \\ 3^{-} & 4^{+} & 2^{-} \\ 0^{+} & 4^{-} & -2^{+} \end{bmatrix}$

try another row/col

this time row 2

$$A = \begin{bmatrix} 2^{+} & 0^{-} & 4^{+} \\ 3^{-} & 4^{+} & 2^{-} \\ 0^{+} & 4^{-} & -2^{+} \end{bmatrix}$$

$$\det A = -(3)\begin{vmatrix} 0 & 4 \\ 4 & -2 \end{vmatrix} + (4)\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} - (2)\begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix}$$

$$= (-3)(-16) + (4)(-4) - (2)(8) = \boxed{16}$$

pick row/col w/ many zeros to make it easier

example

$$A = \begin{bmatrix} 8 & 0 & 0 & 5 \\ 5 & 8 & 3 & -7 \\ 2 & 0 & 0 & 0 \\ 7 & 2 & 1 & 7 \end{bmatrix} \qquad \text{cofactor signs: } \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

"good" choices of row/col : row 3 (has 3 zeros)

expand along row 3

$$\det A = (2)\underbrace{\begin{vmatrix} 0 & 0 & 5 \\ 8 & 3 & -7 \\ 2 & 1 & 7 \end{vmatrix}}_{\text{expand along row 1}} + (0)\begin{vmatrix} \text{I.D.C} \end{vmatrix} + (0)\begin{vmatrix} \text{I.D.C 2} \end{vmatrix} + (0)\begin{vmatrix} \text{I.D.C.3} \end{vmatrix}$$

$$= (2)\left((5)\begin{vmatrix} 8 & 3 \\ 2 & 1 \end{vmatrix}\right) = (10)(8-6) = \boxed{20}$$

Cramer's Rule

another way to solve system (other ways: row reduction
find inverse)

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$\longrightarrow \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\vec{b}}$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

← det of A w/ 1st col replaced by $\vec{b}$

← det A

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

← det of A w/ 2nd col replaced by $\vec{b}$

← det A

$$x_1 + 3x_2 = 9$$
$$2x_1 + x_2 = 8$$

seen this before. $x_1 = 3$, $x_2 = 2$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 9 & 3 \\ 8 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{-15}{-5} = 3$$

$$x_2 = \frac{\begin{vmatrix} 1 & 9 \\ 2 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{-10}{-5} = 2$$

Same idea for 3×3 and beyond

example

$$\begin{aligned} x_1 + x_2 &= 4 \\ -4x_1 + 3x_3 &= 0 \\ x_2 - 3x_3 &= 3 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -4 & 0 & 3 \\ 0 & 1 & -3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$$

find x_1

$$x_1 = \frac{\begin{vmatrix} 4 & 1 & 0 \\ 0 & 0 & 3 \\ 3 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 \\ -4 & 0 & 3 \\ 0 & 1 & -3 \end{vmatrix}} = \cdots = \frac{1}{5}$$

transpose of a matrix : interchange rows and columns

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}_{2\times 2}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3} \qquad B^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times 2}$$