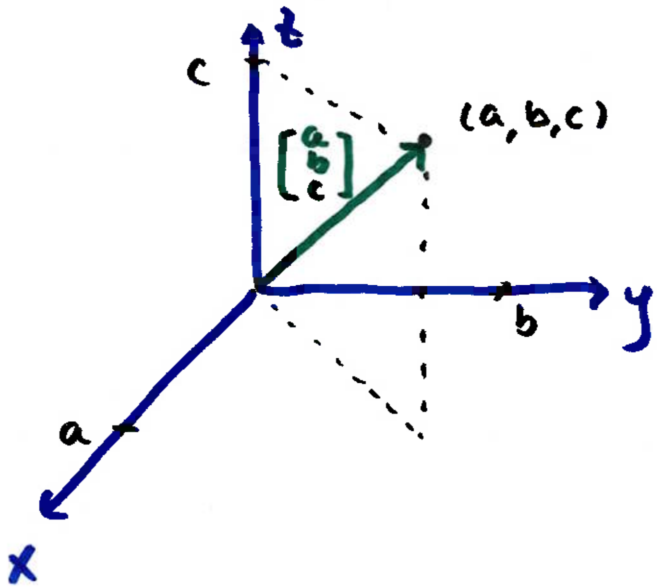
4.1 The Vector Space R^3

3D coordinate system



point (a, b, c)

can be seen as the tip of the vector from origin to the point

$$(a, b, c) \longleftrightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\vec{i} + b\vec{j} + c\vec{k}$$

this is (part of) why we can add/subtract or multiply by scalar w/ vectors

(each component is a number and we can do all the above w/ numbers)

all possible vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ are contained in the <u>vector space R^3</u>

inside R^3 there is R^2 (xy-plane)

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

two R^n vectors are linearly dependent if one is a scalar multiple of the other

$\vec{u}, \vec{v}$ are two R^n vectors

linearly dependent if $\boxed{\vec{u} = c\vec{v} \quad \text{or} \quad \vec{v} = d\vec{u}}$

example: $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

$-2\vec{u} = \vec{v}$ so linearly dependent

if not linearly dependent, then they are linearly independent

$\vec{u} \neq c\vec{v}$

or, $a\vec{u} + b\vec{v} = \vec{0}$ implies $a = b = 0$ only

for example, $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

they are linearly independent because $a\vec{i} + b\vec{j} = \vec{0}$

$\begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ possible if and only if $a = b = 0$

$\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

clearly, not multiples of ~~each~~ each other $\longrightarrow$ linearly indp

so, $a\vec{u} + b\vec{v} = \vec{0}$ possible if $a = b = 0$

$$a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}}_{A}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

notice $\det A = -2 - 2 \neq 0$

if $\vec{u}$, $\vec{v}$ are indp, then when put in as columns of a matrix A, $\det A \neq 0$

$$A\vec{x} = \vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

A^{-1} does not exist if $\det A = 0$

if cols are indp, then $\det A \neq 0$ so A^{-1} exists

if cols of A are indp then A^{-1} exists three or more vectors are linearly <u>dependent</u> if at least one <u>can be written as a linear combination</u> of the others

$\vec{u}, \vec{v}, \vec{w}$ if <u>dep</u> then $\vec{u} = \underbrace{a\vec{v} + b\vec{w}}_{\text{linear combination}}$

or $\vec{v} = c\vec{u} + d\vec{w}$

and so on

> if not, then they are <u>linearly independent</u>
>
> which implies $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$ if and only if $a = b = c = 0$

for example, $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$a\vec{i} + b\vec{j} + c\vec{k} = \vec{0}$ if and only if $a = b = c = 0$

so, $\vec{i}, \vec{j}, \vec{k}$ are linearly indp

$\vec{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

are they linearly indp?

if not, express one as a linear combination of the others

let's try to express $\vec{u}$ as a linear combo of $\vec{v}$ and $\vec{w}$

(if not possible, then they must be indp)

$\vec{u} = a\vec{v} + b\vec{w}$ find a, b

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 8 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} 7 & -1 & 4 \\ 8 & 2 & 3 \end{bmatrix}$$ solve to find a, b using, for example, Gaussian elimination

$\longrightarrow \cdots \longrightarrow$ $a = \frac{1}{2}$, $b = -\frac{1}{2}$

so $\vec{u} = \frac{1}{2}\vec{v} - \frac{1}{2}\vec{w}$

so NOT indp.

example

$$\vec{u} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix}$$

indp?

express one as linear combo of other two?

linearly indp if $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0} \longrightarrow a = b = c = 0$

$$\begin{bmatrix} 4 & -5 & 0 \\ 0 & 1 & -4 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Indp if A^{-1} exists $\Leftrightarrow \det A \neq 0 \rightarrow$ does not tell us a, b, c if the vectors are not indp

but Gaussian elimination can give us a, b, c even if not indp

$$\begin{bmatrix} 4 & -5 & 0 & 0 \\ 0 & 1 & -4 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

row ops

$$\rightarrow \cdots \rightarrow \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

↳ existence of free variables → infinitely-many solutions

↓

$a=b=c=0$ is NOT the only solution

so, $\vec{u}, \vec{v}, \vec{w}$ are NOT indp

now write one as linear combo of other two

$$\begin{array}{cccc} a & b & c & \\ \end{array}$$

$$\begin{bmatrix} \boxed{1} & -1 & -1 & 0 \\ 0 & \boxed{1} & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c is free → $c = r$

row 2: $b - 4c = 0 \rightarrow b = 4r$

row 1: ... $a = 5r$

$a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$

$5r\vec{u} + 4r\vec{v} + r\vec{w} = 0$ choose $r=1$

$5\vec{u} + 4\vec{v} + \vec{w} = 0$ or $\boxed{\vec{w} = -5\vec{u} - 4\vec{v}}$

or $\vec{u} = -\frac{4}{5}\vec{v} - \frac{1}{5}\vec{w}$