1.2 Integrals as General and Particular Solutions

first-order differential eq: $\frac{dy}{dx} = \underbrace{f(x, y)}_{\text{generally contain } x \text{ and } y}$

independent variable (x)

dependent (y)

e.g. $\frac{dy}{dx} = xy^2$ solution: $y = ?$

we will start with the case where right side contains no y

$$\frac{dy}{dx} = f(x) \rightarrow \text{just calculus}$$

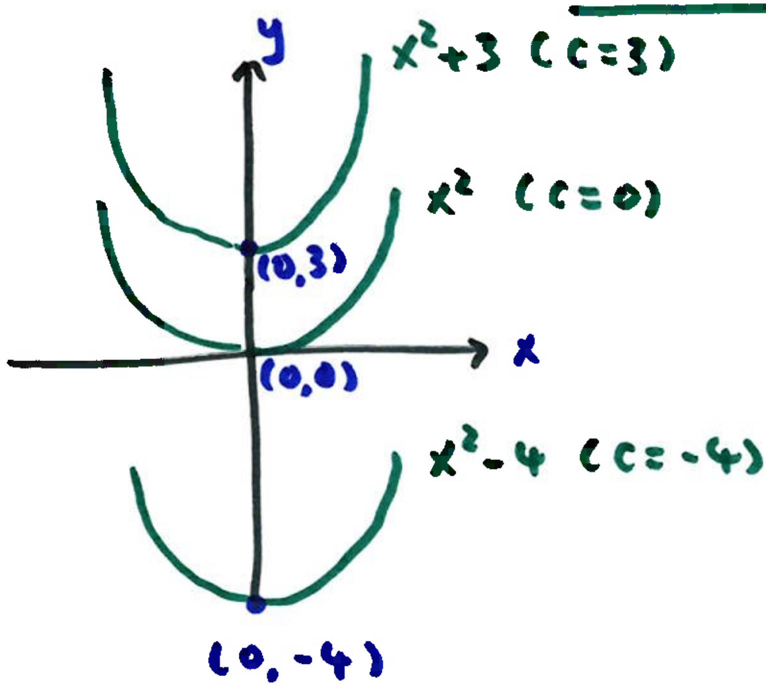
$$y = \int f(x)\, dx$$

e.g. $\frac{dy}{dx} = 2x$

$$y = \int 2x\,dx = \underbrace{x^2 + C}$$

infinitely many solutions

→ <u>General solution</u>



altogether they are the <u>general solution</u>

each one is called a <u>particular solution</u>

finding C is equivalent to specifying a point the solution must go through. For example, solution to

$$\frac{dy}{dx} = 2x, \quad \boxed{y(0) = 3}$$ → initial or side condition

Solution is $y = x^2 + 3$

Solution curves do <u>not</u> have common points

<u>example</u>

$$\frac{dy}{dx} = x\sqrt{x^2+1} \qquad y(0) = 2$$

initial condition given → "initial value problem" or IVP

$$y = \int x\sqrt{x^2+1}\,dx$$

$u = x^2 + 1$

$du = 2x\,dx$

$$= \int \frac{1}{2} u^{1/2}\,du = \frac{1}{2}\,\frac{u^{3/2}}{2/3} + C = \frac{1}{3}u^{3/2} + C$$

$$y = \frac{1}{3}(x^2+1)^{3/2} + C \quad \text{(general solution)}$$

use the given initial condition $y(0)=2$ to find C

$$2 = \frac{1}{3}(0+1)^{3/2} + C \quad \text{so} \quad C = \frac{5}{3}$$

$$\boxed{y = \frac{1}{3}(x^2+1)^{3/2} + \frac{5}{3}}$$

second-order : $\dfrac{d^2y}{dx^2} = f(x)$

$$\frac{dy}{dx} = \int f(x)\,dx + C_1 = F(x) + C_1$$

$$y = \int F(x)\,dx + C_1 x + C_2$$

Two constants

Two initial/side conditions to find constants

n^{th}-order $\rightarrow$ n initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad \text{etc}$$

<u>example</u> $\quad y'' = -10$

$$y' = -10x + c_1$$

$$y = -5x^2 + c_1 x + c_2$$

need two conditions : for example $\quad y(0) = 0$

$$y'(0) = 1$$

$$1 = -10(0) + c_1 \rightarrow c_1 = 1$$

$$y = -5x^2 + x + c_2$$

$y(0) = 0 \rightarrow 0 = -5(0) + (0) + c_2$ so $c_2 = 0$

$$\boxed{y = -5x^2 + x}$$

$y'' = -10$ can be interpreted physically

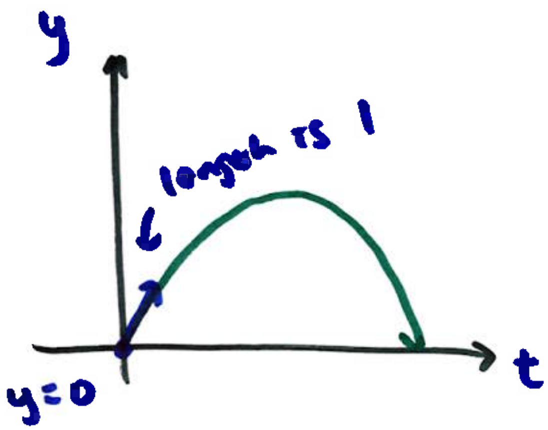
let's use t instead of x as the independent variable

if y is the position of an object

then y' is velocity and y'' is the acceleration

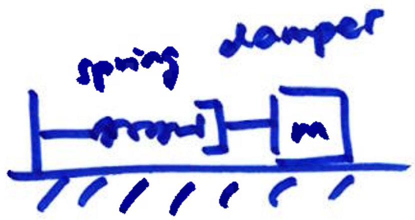
and $y(0) = 0 \rightarrow$ initial position: at $t=0$, y is at 0

$y'(0) = 1 \rightarrow$ initial velocity: at $t=0$, y' is 1



$$y'' + 2y' + y = 0$$

can be interpreted physically as a mass-spring-damper problem

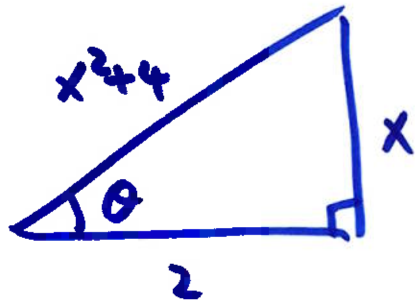


calculus review

trig subs.

$$\frac{dy}{dx} = \frac{1}{x^2+4}$$

triangle with sides x^2+4, x, 2



relate x and θ: $\tan\theta = \frac{x}{2}$

$x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$

$$y = \int \frac{1}{x^2+4}\,dx$$

$$= \int \frac{1}{\underbrace{4\tan^2\theta+4}_{4(\tan^2\theta+) = 4\sec^2\theta}} 2\sec^2\theta\,d\theta = \int \frac{1}{4\sec^2\theta}\cdot 2\sec^2\theta\,d\theta = \int \frac{1}{2}\,d\theta$$

$$y = \frac{1}{2}\theta + c$$

$$\tan\theta = \frac{x}{2} \rightarrow \theta = \tan^{-1}\left(\frac{x}{2}\right)$$

$$\boxed{y = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c}$$