7.1 First-order system of Diff. Eqs.

$$x'(t) = f(x, y, t)$$
$$y'(t) = g(x, y, t)$$

x, y : dependent variables

t : independent variable

for example,

$$x' = y$$
$$y' = -x$$

in general, these equations are coupled → depend on other variable(s)

here, to solve for x, we need to know y and vice versa.

one way to solve a system like this is to convert it into a 2nd-order eq.

$x' = y$

$y' = -x$

differentiate: $x'' = y'$

then from $y' = -x$, we get $x'' = -x$

or $x'' + x = 0$ this we can solve

$r^2 + 1 = 0 \quad r = \pm i$

so, $\boxed{x(t) = c_1 \cos t + c_2 \sin t}$

then from $x' = y$ we get

$\boxed{y = x'(t) = -c_1 \sin t + c_2 \cos t}$

in general, n 1st-order $\rightarrow$ one n^{th}-order

and vice versa. (n^{th}-order $\rightarrow$ n 1st-order)

<u>example</u> $x'' + 3x' + 7x = t^2$

2nd-order → system of 2 1st-order

define two variables to represent

$\underbrace{x \text{ and } x'}_{\text{all dependent variables below the highest order of derivative}}$

let $x_1 = x$

$x_2 = x'$

1st eq. of the system: $\boxed{x_1' = x_2}$

2nd eq of the system: rewrite the given diff. eq. in terms of x_2'

$$x'' + 3x' + 7x = t^2$$

$$x_2' + 3x_2 + 7x_1 = t^2$$

So,
$$\boxed{x_2' = -3x_2 - 7x_1 + t^2}$$

<u>example</u>
$$x^{(4)} + 6x''' - 3x'' + x' + 10x = \cos 3t$$

4th-order $\rightarrow$ 4 1st-order

define 4 variables to represent x, x', x'', x'''

(all derivs below the highest one)

$x_1 = x$

$x_2 = x'$

$x_3 = x''$

$x_4 = x'''$

write a differential eq. for each

$$x_1' = x' = x_2$$

$$\boxed{x_1' = x_2}$$

$$\boxed{x_2' = x_3}$$

$$\boxed{x_3' = x_4}$$

last one: the original diff. eq. rewritten in terms of the new variables

$$x^{(4)} + 6x''' - 3x'' + x' + 10x = \cos 3t$$

$$x_4' + 6x_4 - 3x_3 + x_2 + 10x_1 = \cos 3t$$

$$\boxed{x_4' = -6x_4 + 3x_3 - x_2 - 10x_1 + \cos 3t}$$

n^{th}-order $\to$ n 1st-order

system of two ~~1st~~ 2nd-order $\to$ 4 1st-order

<u>example</u>

$$x'' - 5x' - 4x + 6y = 0$$
$$y'' + 6y' + 5x + 5y = 0$$

define new variables to represent

$$x, x', y, y'$$

$$z_1 = x$$
$$z_2 = x'$$
$$z_3 = y$$
$$z_4 = y'$$

write diff. eq. for each

$$\boxed{z_1' = z_2}$$

$z_2' = x''$ → look at the 1st 2nd-order eq.

$$x'' - 5x' - 4x + 6y = 0$$

$$x'' = 5x' + 4x - 6y$$

$$z_2' = 5z_2 + 4z_1 - 6z_3$$

$$\boxed{z_2' = 5z_2 + 4z_1 - 6z_3}$$

$$\boxed{z_3' = z_4}$$

$z_4' = y''$ → the other 2nd-order eq.

$$y'' + 6y' + 5x + 5y = 0$$

$$y'' = -6y' - 5x - 5y$$

$$\boxed{z_4' = -6z_4 - 5z_1 - 5z_3}$$

we can rewrite the system as a matrix eq.

$$\underbrace{\begin{bmatrix} z_1' \\ z_2' \\ z_3' \\ z_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4 & 5 & -6 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 0 & -5 & -6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}}_{*\ \vec{z}\,' = A\vec{z}}$$

$$\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

Solution of the system must have something to do the matrix A

we can see the connection in very system systems

(more details in the rest of ch. 7)

$x'' - x = 0 \longrightarrow x(t) = c_1 e^t + c_2 e^{-t}$

$\boxed{r^2 - 1 = 0} \rightarrow r = 1, -1$

characteristic eq.

turn into a system: let $z_1 = x$

$z_2 = x'$

$$\left.\begin{aligned} z_1' &= z_2 \\ z_2' &= z_1 \end{aligned}\right\} \quad \begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

eigenvalues of A: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \qquad \boxed{\lambda^2 - 1 = 0}$$

characteristic eq.

$\lambda = 1, -1$

$e^t \qquad e^{-t}$